State Math Contest - Junior Exam

Instructions:

- Do not turn this page until your proctor tells you.

- Enter your name, grade, and school information following the instructions given by your proctor.

- This is a multiple choice test with 40 questions. Each question is followed by answers marked a), b), c), d), and e). Only one answer is correct.

- Mark your answer to each problem on the bubble sheet Answer Form with a #2 pencil. Erase errors and stray marks. Only answers properly marked on the bubble sheet will be graded.

- SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.

- You will have 2 hours and 30 minutes to finish the test.

1. Find the slope of the line containing the two points $(2, -4)$ and $(5, 3)$.

    a) $\dfrac{7}{3}$ b) $\dfrac{3}{7}$ c) $\dfrac{-3}{7}$ d) $\dfrac{-7}{3}$ e) $\dfrac{4}{3}$
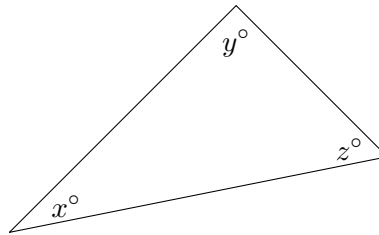
2. Given a point P inside of a square, the distances from P to the sides of the square are 1, 3, 7, and 9. Find the area of the square.

    a) 66   b) 100   c) $100\sqrt{2}$ d) 144   e) 189

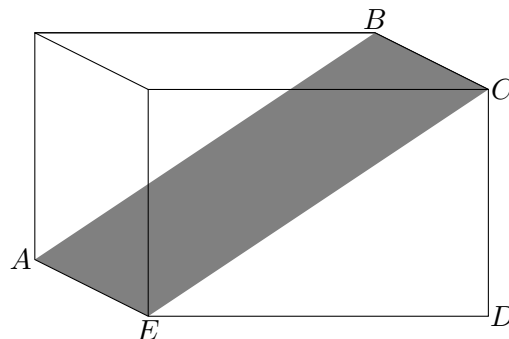3. Compute $2015 + 2015 - 2015 \times |-2015| \div (-2015)$.

    a) 0   b) 2014   c) 2015   d) 4030   e) 6045

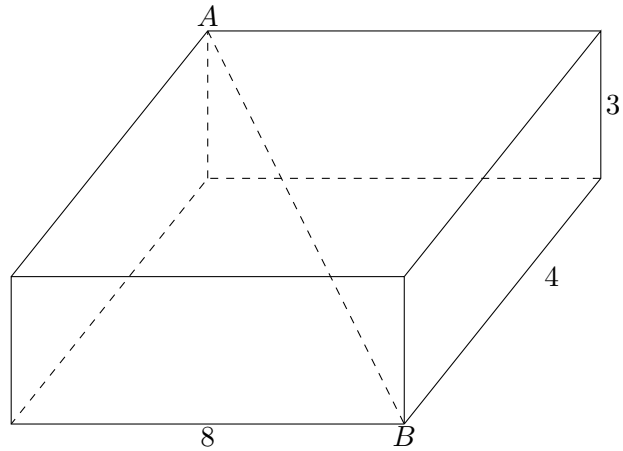4. In the following figure, what is the value of $\dfrac{x+y+z}{15}$?

    a) 5   b) 8

    c) 9   d) 10

    e) 12

5. In the following rectangular prism, AE = 3, DE = 5, and CD = 4. What is the area of the shaded rectangular region?

    a) 9   b) $3\sqrt{41}$

    c) 25   d) $5\sqrt{2}$

    e) $4\sqrt{34}$

6. In the following rectangular prism, what is the length of AB?

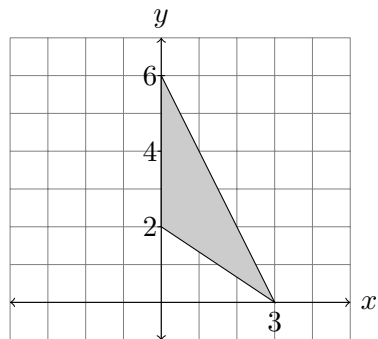   a) $\sqrt{33}$ b) $4\sqrt{6}$

    c)   96   d)   89

    e) $\sqrt{89}$

7. Rotating a triangle in the xy-plane about the y-axis forms a 3-dimensional solid. Find the volume of the solid.

   a) 6π b) 12π

    c) 18π d) 24π

    e) 36π

8. A triangle in the xy-coordinate plane has vertices at $A = (0,0)$, $B = (30,0)$, and $C = (12,6)$. A square is inscribed in the triangle with two vertices on side AB and one vertex each on the other two sides of the triangle. Find the area of the square.

a)   16

b) $\dfrac{81}{4}$

c) $\dfrac{121}{4}$

d) $\dfrac{169}{9}$

e)   25

Questions 9 and 10 refer to the following graph, which represents the cumulative number of miles d that a car has traveled after t minutes.
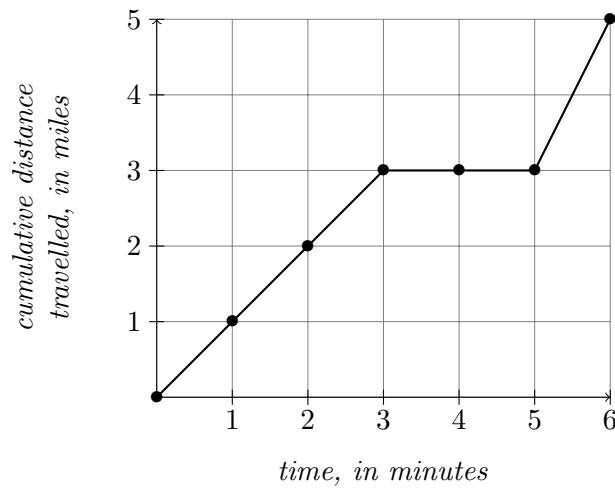
9. The total distance traveled in miles by the car from 0 to 5 minutes is

a)   1   b)   2   c)   3

d)   4   e)   5

10. The average speed of the car from two minutes to six minutes is

a)   2 miles per minute.

b) $\dfrac{3}{4}$ of a mile per minute.

c)   5 miles per minute.

d)   1 mile per minute.

e) $\dfrac{1}{3}$ of a mile per minute.

11. A coin is flipped 8 times. What is the probability that the number of heads is strictly greater than the number of tails?

    a) $\dfrac{4}{9}$ b) $\dfrac{1}{2}$ c) $\dfrac{93}{256}$ d) $\dfrac{23}{84}$ e) $\dfrac{61}{84}$

12. A plane slices a cube so that the intersection is a polygonal region. What is the maximum number of sides of the polygonal region?

    a) 3   b) 4   c) 5   d) 6   e) 7

13. If the repeating decimal $0.151515\ldots$ is converted to a fraction and written in simplest form, what is the numerator?

    a) 3   b) 5   c) 15   d) 25   e) 225

14. I have letters to four people, and envelopes addressed to the four people, and (without looking) I randomly put a letter into each of the four envelopes. What is the probability that none of the letters will get put into its correct envelope?

    a) $\dfrac{1}{4}$ b) $\dfrac{3}{4}$ c) $\dfrac{81}{256}$ d) $\dfrac{3}{8}$ e) $\dfrac{23}{24}$

15. How many numbers in the range from 1 to 1,000, inclusive, are not divisible by the first three primes (2, 3 or 5)?

    a) 166   b) 266   c) 299   d) 701   e) 734

16. If x is in the domain of the function $f(x) = \dfrac{(x^2 - 1)\sqrt{(3x - 2)}}{x - 1}$, then

    a) $0 < x < 2$ or $2 < x < \infty$

    b) $-\infty < x < 1$ or $1 < x < \infty$

    c) $\dfrac{2}{3} \leq x$

    d) $\dfrac{2}{3} \leq x < 1$ or $1 < x < \infty$

    e) x is any real number.

17. The expression $\dfrac{\sqrt{x + 4} - \sqrt{x}}{4}$ is the same as

    a) $\dfrac{1}{\sqrt{x + 4} + \sqrt{x}}$ b) $\dfrac{4}{\sqrt{x + 4} + \sqrt{x}}$ c) $\dfrac{1}{2}$

    d) $\dfrac{1}{\sqrt{x + 4} - \sqrt{x}}$ e) $\dfrac{2x + 4}{\sqrt{x + 4} - \sqrt{x}}$

18. If we are to choose real numbers a and b such that $0 < a < b$, then we can best describe the solution set of $|x + 12| < |3x - 1|$ as the set of all x such that

    a) $-a < x < b$ b) $-b < x < a$

    c) $x < -a$ or $x > b$ d) $x < -b$ or $x > a$

    e) $x < a$ or $x > b$

19. Find the inverse function $f^{-1}(x)$ if $f(x) = \dfrac{2x - 1}{x + 3}$.

    a) $f^{-1}(x) = \dfrac{3x - 1}{2 - x}$ b) $f^{-1}(x) = \dfrac{3x + 1}{2 - x}$

    c) $f^{-1}(x) = \dfrac{2x + 1}{3 - x}$ d) $f^{-1}(x) = \dfrac{x + 1}{2 + 2x}$

    e) $f^{-1}(x) = \dfrac{2 - x}{1 + 3x}$

20. Train stations A and B are on the same railroad line and are 50 miles away from each other. A train leaves station A heading towards station B at 1:00 pm going 20 miles an hour. Another train leaves station B heading toward station A at 2:00 pm going 10 miles an hour. When will the two trains meet each other?

    a)   2:30 pm   b)   3 pm   c)   3:30 pm

    d)   4 pm   e)   4:30 pm

21. If $x^a x^b = 1$ and $x > 1$, find $4a - b^2 + a^2 + 4b - 10$.

    a)   -20   b)   -10   c)   0   d)   10   e)   20

22. Find the number of digits in the product $25^{25} \times 2^{60}$.

    a)   76   b)   37   c)   54   d)   28   e)   65

23. Positive integers m and n satisfy the equation $(2m - 7)(2n - 7) = 25$. What are all possible values for $m + n$?

    a)   2, 20, 24   b)   3, 12, 16   c)   2, 10, 16

    d)   12, 20   e)   2, 12, 20

24. Three sides of a quadrilateral have lengths 3, 4, and 9. There exist positive real numbers a and b such that if l is the length of the fourth side of the quadrilateral, then $a < l < b$, and if l satisfies $a < l < b$, then there exists a quadrilateral with side lengths 3, 4, 9, and l. Find $a + b$.

    a)   18   b)   16   c)   9   d)   32   e)   24

25. Two square regions A and B each have area 8. One vertex of square B is the center point of square A. Find the area of $A \cup B$.

a)   16 b)   15 c) $10\sqrt{2}$

d)   14 e)   Cannot   be determined.

26. A plane slices a cone parallel to the base and one-third the distance from the vertex to the base and a second parallel plane slices the cone two-thirds the distance from the vertex to the base. What fraction of the volume of the cone is between the two slices?

a) $\dfrac{8}{9}$ b) $\dfrac{7}{27}$ c) $\dfrac{1}{3}$ d) $\dfrac{1}{9}$ e) $\dfrac{2}{27}$

27. Suppose that the numbers 4-9, inclusive, are arranged in three pairs of distinct numbers. The numbers in each pair are added together, and the resulting three numbers are then multiplied together. What is the maximum value of the resulting product?

a)   less than 1,801 b)   between 1,801 and 1,900, inclusive

c)   between 1,901 and 2,000, inclusive d)   between 2,001 and 2,100, inclusive

e)   more than 2,100

28. Suppose that Miles lists all possible (distinct) rearrangements of the letters in the word MATHE-MATICS. He then picks one rearrangement at random. What is the probability that the first five letters of this rearrangement are ATTIC (in order)?

a) $\dfrac{1}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$ b) $\dfrac{2}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$

c) $\dfrac{4}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$ d) $\dfrac{6}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$

e) $\dfrac{8}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$

29. The Francis family has 5 children: Amanda, Andrea, Braden, Bethany, and Caitlyn. Mr. Francis brings cookies home one day. He has 2 sugar cookies, 2 peanut butter cookies, and one oatmeal cookie. In how many ways can Mr. Francis distribute the cookies to his children if each child must have a whole cookie, Amanda and Andrea insist on having the same type of cookie, and Braden and Bethany refuse to eat the same kind of cookie?

a)   4 b)   5 c)   6 d)   7 e)   8

30. Annabeth and Elinor are hoping to meet for dinner. They will each arrive at their favorite restaurant at a random time between 6:00 and 8:00 pm, stay for 20 minutes, and leave. What is the probability that they will see each other at the restaurant?

a) $\dfrac{7}{36}$ b) $\dfrac{9}{36}$ c) $\dfrac{11}{36}$ d) $\dfrac{13}{36}$ e) $\dfrac{15}{36}$

31. Let S be the set of all positive integers n such that n^3 is a multiple of both 16 and 24. What is the largest integer that is a divisor of every integer n in S?

    a) 6   b) 12   c) 18   d) 24   e) 216

32. Tyson is three times as old as Mandi. Two years ago, Tyson was four times as old as Mandi. How old is Mandi now?

    a) 16   b) 14   c) 7   d) 6   e) 4

33. Suppose that
$$A \star B = 2A + 3B, \text{ and } C\heartsuit D = \frac{C^2 + 2D}{D}.$$
What is $(12 \star 4)\heartsuit 3$?

    a) 432   b) 433   c) 434   d) 435   e) 436

34. Solution 1 contains only liquids a and b in a ratio of $1 : 4$. Solution 2 contains also contains only liquids a and b, but in a ratio of $1 : 1$. Solution 3 is obtained by mixing Solutions 1 and 2 in a ratio of $5 : 1$. How many Tablespoons of liquid a are in 60 Tablespoons of Solution 3?

   a) 15   b) 18   c) 20   d) 24   e) 30

35. Suppose that in the following figure, $AB = BC = CD = DE = EF$. What is the ratio of the area of triangle ABH to the area of triangle AFG?
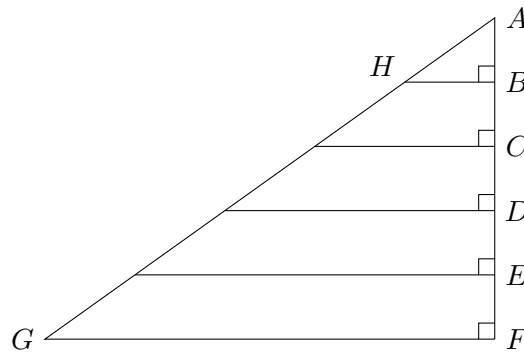

    a) 1:5   b) 1:10   c) 1:15   d) 1:20   e) 1:25

36. If the average test score for five students is 92, which of the following is the highest score a sixth student could get so that the average of all six scores would be no more than 86?

    a) 55   b) 56   c) 57   d) 58   e) 59

37. How many odd four-digit numbers are there that do not contain the digit 6?

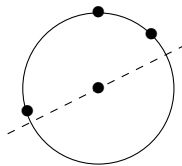   a) 2560   b) 3240   c) 3645   d) 4050   e) 5000

38. Points A, B, and C lie on the circle. The point D is the center of a circle and lies on the line segment AC. If $AB = 6$ and $BD = 5$, find BC.
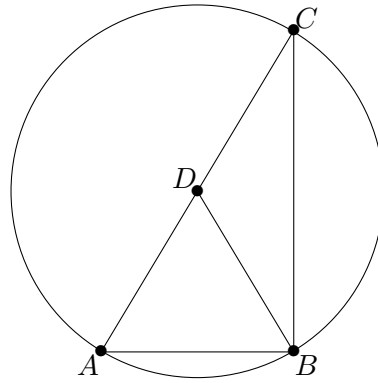
   a) 6   b) 6.5   c) 7   d) 7.5   e) 8

39. How many isosceles triangles can be drawn if each vertex must be one of the dots in the following square lattice?
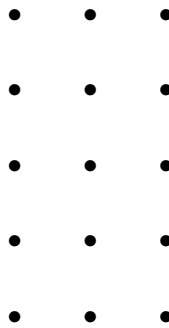
   a)   at most 80   b)   between 81 and 90, inclusive

   c)   between 91 and 100, inclusive   d)   between 101 and 110, inclusive

   e)   greater than 110

40. If three points are scattered randomly on a circle, what is the probability that one can draw a line through the center of the circle, such that all three points lie on one side of the line?

   a) $\dfrac{1}{2}$ b) $\dfrac{3}{4}$ c) $\dfrac{7}{8}$ d) $\dfrac{5}{8}$ e) $\dfrac{2}{3}$

END OF EXAM