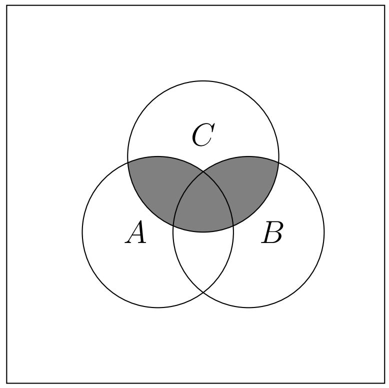
Contents

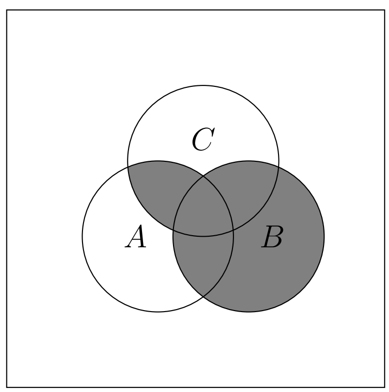
1 Venn Diagrams

1. Venn diagrams represent the relationship between sets. Below is the Venn diagram for three sets, A, B, and C. The shaded region represents the union of the two intersections between A and C, and B and C.

$$(A \cap C) \cup (B \cap C)$$



Which expression represents the shaded area of the Venn diagram?



A. $(A \cap C) \cup B$

B. $A \cup B$

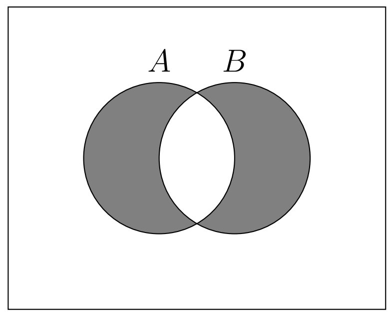
C. $A \cap B \cap C$

D. $(A \cap C) \cup (B \cup C)$

E. $(A \cap C) \cap (B \cup C)$

2. The superscript, ^{C}, refers to the complement of a set. For example, in the Venn diagram below the shaded region represents the complement of the two intersections between A and B intersecting $A \cup B$.

$$(A \cap B)^C \cap (A \cup B)$$



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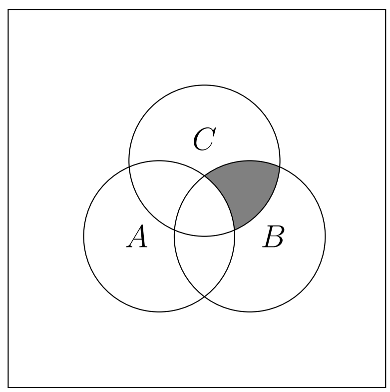


A. $(B \cap A) \cup (B \cap C)$

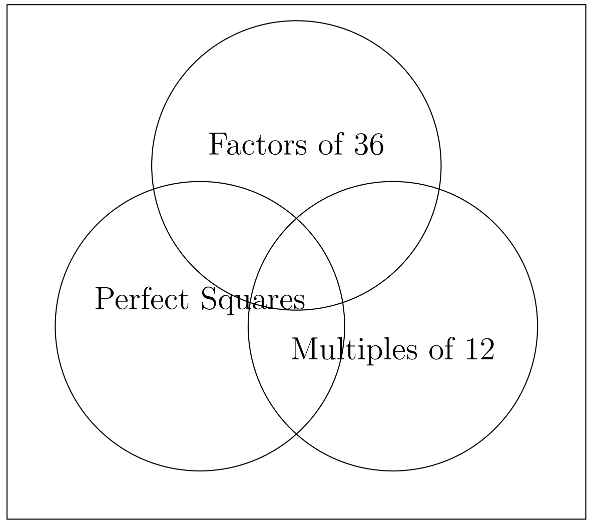
B. $(B \cup A)^C \cup (B \cap A)$

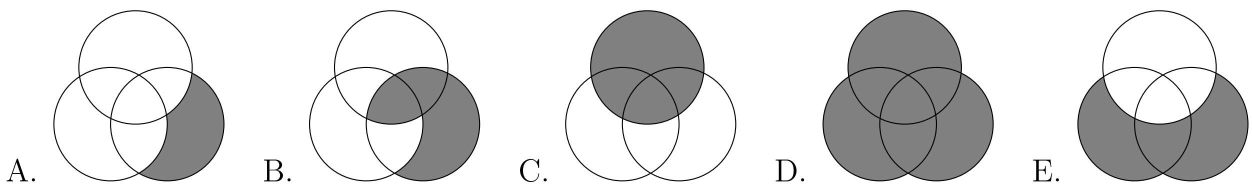
C. $C^C \cap (B \cap A)^C$

D. $A^C \cap (B \cap C)$

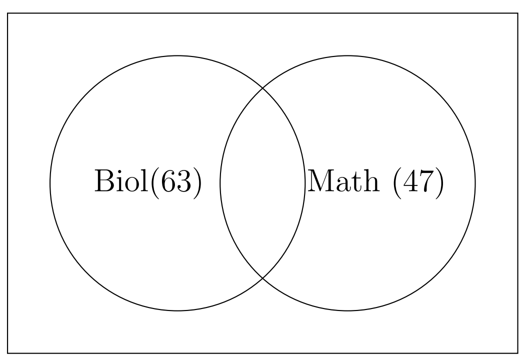
E. $B^C \cap (A \cup C)$

3. Which diagram depicts the location of 24?





4. A group of mathematicians and biologists assembled at a conference. Of the 100 professionals in attendance, 63 were biologists and 47 were mathematicians. How many were mathematical-biologists?



A. 8

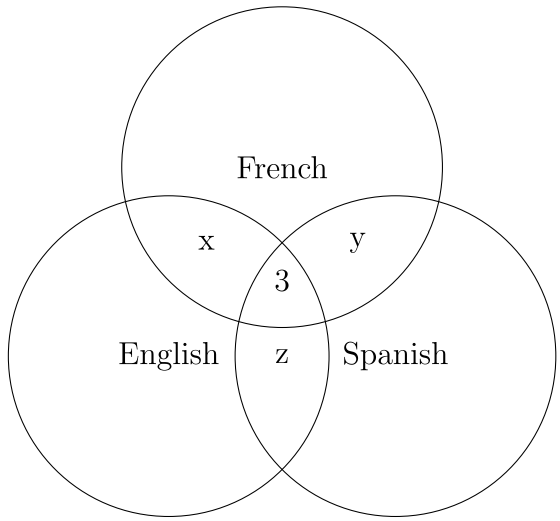
B. 18

C. 24

D. 10

E. 12

5. A survey showed that among a certain set of 43 people, only 3 spoke all three languages, English, Spanish, and French, while 13 spoke exactly two of these languages and 27 spoke just one of these languages. A total of 33 people spoke English. Also a total of 10 spoke French, but each of these also spoke English or Spanish, or both. How many of the 43 people spoke Spanish. The following Venn diagram might help in setting up a system of equations.



A. 15 B. 16. C. 17 D. 18 E. 19

2 Continued Fractions

6. Continued fractions are of the form

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}}}}$$

This pattern might terminate, or continue forever. The clumsy expression above can be represented as a list (or sequence) of the form

$$[a_0; a_1, a_2, a_3, a_4, \dots]$$

Notice that a_0 could be zero, but the rest of the elements are non-zero. For example,

$$\frac{1}{8} = 0 + \frac{1}{7 + \frac{1}{1}} = [0; 7, 1]$$

and

$$\frac{5}{8} = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}}} = [0; 1, 1, 1, 1, 1]$$

Which is the continued fraction for $\frac{5}{7}$?

A. $[0; 1, 7, 5]$

B. $[0; 1, 5, 1]$

C. $[0; 2, 2, 1]$

D. $[0; 1, 2, 1]$

E. $[0; 1, 2, 2]$

7. Recursive relationships can lead to continued fractions. For example, by repeated substitutions, the relationship

$$x = 1 + \frac{1}{x}$$

leads to the continued fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}$$

Use the relationship

$$\sqrt{x} = 1 + \frac{x-1}{1+\sqrt{x}}$$

to find the list for the continued fraction corresponding to $\sqrt{2}$

A. $[1; 2, 2, 2, 2, 2 \ldots]$

B. $[1; 2, 3, 4, 5, 6 \ldots]$

C. $[1; 4, 1, 4, 1, 4 \ldots]$

D. $[1; 2, 2, 3, 3, 4 \ldots]$

E. $[1; 2, 1, 2, 1, 2 \ldots]$

8. Given

$$x = a + \cfrac{1}{a + \cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{\ddots}}}},$$

which of the following is also equal to x?

A. $\sqrt{a}$

B. $\dfrac{a+\sqrt{a}}{2}$

C. $\dfrac{\sqrt{a^2+4}}{2}$

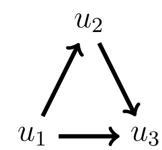
D. $a + \sqrt{a^2+4}$

E. $\dfrac{a+\sqrt{a^2+4}}{2}$

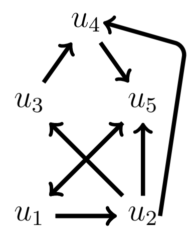
3 Graph Theory

9. Below is the adjacency matrix associated with the digram of graph, D_1.

$$A(D_1) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



Which of the following adjacency matrices is associated with the graph D_2 whose digram given to the right?



A. $\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

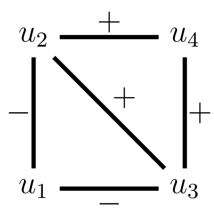
B. $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

C. $\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$

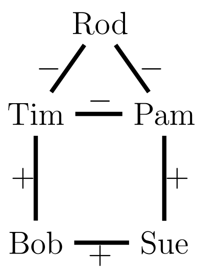
D. $\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

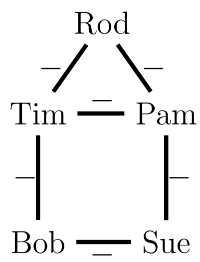
E. $\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

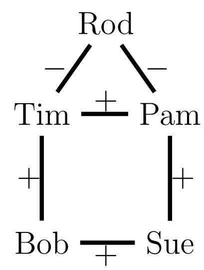
10. A signed graph has a value assigned to each edge of the graph. In the illustration below, the edges with a minus sign have a negative value and the edges with a positive sign have a positive value. This describes the relationship between the nodes. The example below is termed a *balanced* signed graph, because the product of the signs around any closed loop is positive.

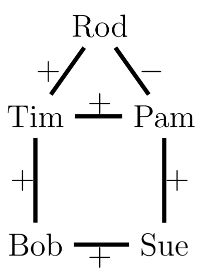


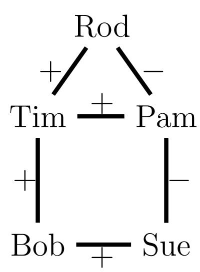
Five people have been selected to work together on a project. Not everyone likes each other, but some people get along well. The sign indicates who works well together, a positive meaning people work well, and a negative meaning they don't work well. The outcome of the group depends on the group dynamic being balanced. Which of the following group dynamics would result in successful completion of the project?

A. 

B. 

C. 

D. 

E. 

11. A railroad baron wishes to connect four rural villages, named A, B, C, and D, by rail. He had his surveyors investigate the best possible route between every pair of villages, and provide a cost estimate for each potential route, in gold bars:

from \ to	A	B	C	D
A	0	13	5	14
B	13	0	6	12
C	5	6	0	7
D	14	12	7	0

What is the minimum cost in gold bars to connect all the villages by rail?

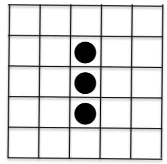
A. 16 bars

B. 21 bars

C. 26 bars

D. 18 bars

E. None of the above.



12. The Game of Life is a cellular automaton devised by the British mathematician John Horton Conway in 1970. The universe of the Game of Life is an infinite two-dimensional orthogonal grid of square cells, each of which is in one of two possible states, alive or dead. Every cell interacts with its eight neighbours, which are the cells that are horizontally, vertically, or diagonally adjacent. At each step in time, the following transitions occur:

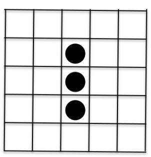
    i. Any live cell with fewer than two live neighbours dies, as if caused by under-population.

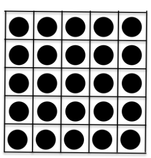
    ii. Any live cell with two or three live neighbours lives on to the next generation.

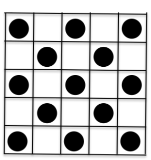
    iii. Any live cell with more than three live neighbours dies, as if by overcrowding.

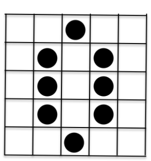
    iv. Any dead cell with exactly three live neighbours becomes a live cell, as if by reproduction.

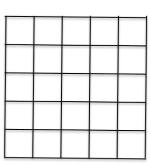
    Given the initial condition in the figure above, in which black dots indicate live cells, what is the configuration after 2 generations?

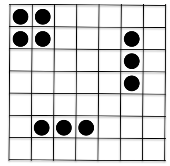
    A. 

    B. 

    C. 

    D. 

    E. 



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    i. Any live cell with fewer than two live neighbors dies, as if caused by under-population.

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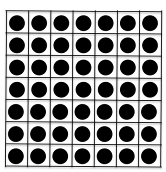
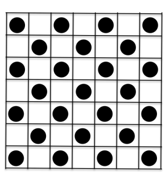
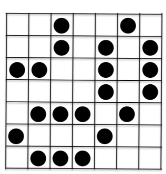
    iii. Any live cell with more than three live neighbors dies, as if by overcrowding.

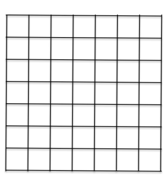
    iv. Any dead cell with exactly three live neighbors becomes a live cell, as if by reproduction.

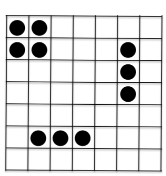
    Given the initial condition in the figure above, in which black dots indicate live cells, what is the configuration after 6 generations?

    A. 

    B. 

    C. 

    D. 

    E. 

4 Special Functions and Series

14. The Riemann zeta function, $\zeta(x)$, is defined

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \ x > 1.$$

What is $\zeta(x)(1 - 2^{-x})$?

A. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)^x}$

B. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)^x}$

C. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^x}$

D. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^x}$

E. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^x}$

15. The Gamma function, $\Gamma(t)$, $t \in \mathbb{R}, t \neq 0$, is defined by

$$\Gamma(t+1) = t\Gamma(t), \ t \neq -n, n \in \mathbb{N}.$$

With the special case $\Gamma(1/2) = \sqrt{\pi}$, what is $\Gamma(-3/2)$?

A. 1

B. $\dfrac{\sqrt{\pi}}{2}$

C. π

D. $-\dfrac{\sqrt{\pi}}{4}$

E. $\dfrac{4\sqrt{\pi}}{3}$

16. The Psi function, $\psi(x)$, $x \neq 0, -1, -2, \ldots$, is defined as the logarithmic derivative of the Gamma function. From this one arrives at a noteworthy special value

$$\psi(1) = \frac{\Gamma'(1)}{\Gamma(1)} = -\gamma$$

where γ is the *Euler Mascheroni constant* (a constant real number). Application of the derivative yields the important relationship

$$\psi(x+1) = \psi(x) + \frac{1}{x}$$

In the case of $x = n \in \mathbb{N}$ we have

$$\psi(n+1) = \psi(1) + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = -\gamma + \sum_{k=1}^{n} \frac{1}{k},\ n = 1, 2, 3, \ldots$$

Use these properties to sum the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

A. $\frac{13}{20}$

B. $\frac{3}{4}$

C. $\frac{17}{25}$

D. $\frac{7}{10}$

E. $\frac{4}{5}$

17. A mathematician named Appell developed a notation that is closely related to the Gamma function and used in analyzing Hypergeometric functions. Appell's notation is defined by the following

$$\begin{aligned}(a,0) &= 1\\ (a,n) &= a(a+1)(a+2)\cdots(a+n-1), n=1,2,3,\ldots\\ (a,-n) &= \frac{1}{(a+1)(a+2)\cdots(a+n-1)}, a\neq -1,-2,-3,\ldots,\ n=1,2,3,\ldots\end{aligned}$$

It follows from this definition that

$$(a,n)=\frac{\Gamma(a+n)}{\Gamma(a)}$$

For example we see that

$$(3,5)=3\cdot 4\cdot 5\cdot 6\cdot 7=2520,$$

and alternatively

$$(3,5)=\frac{\Gamma(3+5)}{\Gamma(3)}=\frac{(8-1)!}{(2-1)!}=\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}{1\cdot 2}=3\cdot 4\cdot 5\cdot 6\cdot 7=2520,$$

What is the value of $(-2,4)$?

A. -1

B. 1

C. 2

D. 0

E. 3

18. Considering Appell's notation from the previous problem,

$$\begin{aligned}(a,0) &= 1\\ (a,n) &= a(a+1)(a+2)\cdots(a+n-1), n=1,2,3,\ldots\end{aligned}$$

what is $(-a,n)$ for $a=1,2,3,\ldots$?

A. $(-a,n) = \begin{cases} \frac{1}{(a-n)!} & : 0 \le n \le a \\ 0 & : n > a \end{cases}$

B. $(-a,n) = \begin{cases} (-1)^n \frac{1}{(a-n)!} & : 0 \le n \le a \\ 0 & : n > a \end{cases}$

C. $(-a,n) = \begin{cases} (-1)^n \frac{a!}{n!} & : 0 \le n \le a \\ 0 & : n > a \end{cases}$

D. $(-a,n) = \begin{cases} \frac{n!}{a!} & : 0 \le n \le a \\ 0 & : n > a \end{cases}$

E. $(-a,n) = \begin{cases} (-1)^n \frac{a!}{(a-n)!} & : 0 \le n \le a \\ 0 & : n > a \end{cases}$

19. Considering the aforementioned Appell's notation,

$$\begin{aligned}(a,0) &= 1\\ (a,n) &= a(a+1)(a+2)\cdots(a+n-1), n=1,2,3,\ldots\end{aligned}$$

what is $\begin{pmatrix} -a \\ n \end{pmatrix}$ for $a,n=1,2,3,\ldots$?

A. $\dfrac{1}{n!}(a,n)$

B. $\dfrac{(-1)^n}{n!}(a,n)$

C. $-\dfrac{1}{n!}(n,a)$

D. $\dfrac{(-1)^n}{a!}(a,n)$

E. $\dfrac{(-1)^n}{(n-a)!n!}(a,n)$

20. Some theoretical physicists define the infinite series

$$S = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \dots = \frac{1}{2}.$$

The following reasoning helps to illustrate their thinking.

$$1 - S = 1 - (1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \dots)$$

Simplifying returns

$$1 - S = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \dots = S$$

So $1 - S = S$ which by solving for S implies that $S = \frac{1}{2}$. There are more complicated explanations for how string theorists arrive at this conclusion, but if we define

$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 \dots = \frac{1}{2},$$

what does the following series equal?

$$P = \sum_{n=0}^{\infty} (-1)^n (n+1) = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 \dots$$

A. $\frac{1}{10}$

B. $\frac{1}{4}$

C. $\frac{1}{8}$

D. $\frac{1}{6}$

E. $\frac{1}{12}$

5 Logic

21. A statement can be true or false, denoted 1 for true and 0 for false. For example:

    1. P : "All dogs are mammals." has a value 1.
    2. Q : "All dogs are brown." has a value 0.

    Compound statements are formed from other statements by using the connectives $\wedge$, (and) and $\vee$ (or). The truth of the compound statement will depend on the truth of its components.

    The following tables presents all possible combinations of arbitrary statements P and Q. And the operation, R, displays the results in the final row of the table for the operation $R = P \wedge Q$ and $R = P \vee Q$.

P	0	0	1	1
Q	0	1	0	1
$P \wedge Q$	0	0	0	1

P	0	0	1	1
Q	0	1	0	1
$P \vee Q$	0	1	1	1

What is the operation, R, in the following table?

P	0	0	0	0	1	1	1	1
Q	0	0	1	1	0	0	1	1
S	0	1	0	1	0	1	0	1
R	0	1	1	1	1	1	1	1

A. $Q \vee (P \wedge S)$

B. $Q \wedge (P \vee S)$

C. $P \wedge (Q \vee S)$

D. $Q \vee P \vee S$

E. $(Q \wedge P) \vee (P \wedge S)$

22. The symbol $\neg$ is the negation of the statement. For example $\neg P$ reads "not all dogs are mammals", which would have a value of false, or 0.

    What is the operation, R, in the following table

P	0	0	1	1
Q	0	1	0	1
R	1	1	0	0

A. $\neg Q \vee P$

B. $Q \vee \neg P$

C. $\neg(Q \wedge P) \wedge \neg P$

D. $\neg Q \wedge P$

E. $\neg(Q \wedge P)$

23. The following table presents all possible combinations of arbitrary statements P and Q, and the conditional statement, "If P then Q", $P \Rightarrow Q$.

P	0	0	1	1
Q	0	1	0	1
$P \Rightarrow Q$	1	1	0	1

What is the operation, R, in the following table

P	0	0	1	1
Q	0	1	0	1
R	1	1	0	0

A. $P \Rightarrow \neg Q$

B. $\neg(P \Rightarrow Q)$

C. $\neg P \Rightarrow \neg Q$

D. $(P \Rightarrow Q) \wedge \neg P$

E. $\neg Q \vee (P \Rightarrow \neg Q)$

6 Polynomials

24. Given the pattern in the following equalities,

$$\begin{aligned} 4^2 + 3^2 &= 5^2 \\ 8^2 + 15^2 &= 17^2 \\ 12^2 + 35^2 &= 37^2 \\ 16^2 + 63^2 &= 65^2 \end{aligned}$$

which of the sets below contains the x and y that satisfy $18^2 + x^2 = y^2$.

A. {73, 74, 75, 76, 77}

B. {88, 89, 90, 91, 92 }

C. {93, 94, 95, 96, 97}

D. {83, 84, 85, 86, 87}

E. {78, 79, 80, 81, 82}

25. If $x = 1$ is a solution of the equation $4a + b(x-4)^2 + c(x-4)^4 + d(x-4)^6 = 0$, then another solution is

A. $x = -3$

B. $x = 3$

C. $x = -1$

D. $x = 7$

E. $x = 5$

26. What is the relationship between two distinct positive real numbers a and b such that they satisfy the equation $b^2 + a = a^2 + b$

A. $b = 1 + a$

B. $b = -a$

C. $b = 1 - a$

D. $b = a - 1$

E. $b = 2a$

27. Suppose the two quadratic equations $x^2 - 5x + k = 0$ and $x^2 - 9x + 3k = 0$ have a nonzero root in common. What set contains the value of k?

A. $\{2, 7, 12, 17\}$

B. $\{3, 8, 13, 18\}$

C. $\{6, 11, 16, 21\}$

D. $\{4, 9, 14, 19\}$

E. $\{5, 10, 15, 20\}$

7 Probability

28. A restaurant has an automatic dishwasher that has 6 large pots, 4 small pots, 3 large pot lids, and 5 small pot lids. A cook grabs a pot at random and then a lid at random. What is the probability that the lid will fit the pot?

    A. $\dfrac{\binom{10}{2}+\binom{8}{2}}{\binom{18}{2}}$

    B. $\dfrac{1}{80}$

    C. $\dfrac{9}{40}$

    D. $\dfrac{\binom{9}{3}\binom{9}{2}}{\binom{18}{2}}$

    E. $\dfrac{38}{80}$

29. A small computer store sells laptops that have various combinations of screen types and operating systems. The table below provides the proportions of each combination that the store has available. One laptop is to be selected at random from this store. If the laptop that is selected has a Touch-Screen, what is the probability that the operating system is Windows 8?

LAPTOPS		
	Touch-Screen	No Touch-Screen
Windows 8	0.40	0.20
Other operating system	0.10	0.30

    A. 0.40

    B. 0.30

    C. 0.80

    D. 0.90

    E. None of the above

30. Bill and Bob have agreed to meet for running between 6 pm and 8 pm, but neither of them remember the time of the meeting. If any of the two comes to the park between 6 pm and 8pm and stays for 12 minutes unless his friend shows up before that period of time is up, what is the probability that they will meet to start running between 6 pm and 8 pm? Assume that all random times are equally likely.

    A. 50%

    B. 24%

    C. 25%

    D. 19%

    E. 12%