Utah State Mathematics Contest
Junior Exam Solutions
March 14, 2012

1. What is the units digit of 2012^{2012}?

Solution: (c) The units digit is determined by the power of the last digit 2 alone modulo 4:

1 mod 4	2 mod 4	3 mod 4	0 mod 4
$2^1 = 2$,	$2^2 = 4$,	$2^3 = 8$,	$2^4 = 16$,
$2^5 = 32$,	$2^6 = 64$,	$2^7 = 128$,	$2^8 = 256$,

etc. Because the exponent 2012 is divisible by 4, the last digit will be 6.

2. A train, traveling at a constant speed, takes 20 seconds from the time it enters a tunnel that is 300 meters long until it emerges from the tunnel. A bat sleeping on the ceiling of the tunnel is directly above the train for 10 seconds. How long is the train?

Solution: (d) The train travels 300 m plus its length L in 20 seconds. At the same speed, the train travels its length in 10 seconds. Therefore, the train travels twice its length in 20 seconds so the train's length is 300 m. Because the word "emerges" could be taken in two different ways, we also accepted answer (a), 150 m.

3. Three dice are tossed. What is the probability that the numbers shown will all be different?

Solution: (a) The number of ways only distinct numbers appear is $6 \cdot 5 \cdot 4$ and the total number of possible outcomes is 6^3. The probability of all different numbers is

$$\frac{6 \cdot 5 \cdot 4}{6 \cdot 6 \cdot 6} = \frac{5}{9}.$$

4. An archery target has two scoring areas: one worth 5 points and another worth 7 points. What is the largest score impossible to obtain?

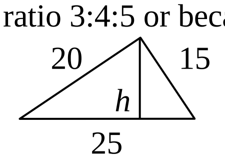
Solution: (d) First, we note that 5 and 7 are relatively prime. (Clearly, if both numbers were even, no odd score would be possible.) In listing the possible scores, once there are 5 consecutive possible scores, all scores after that are possible. Certainly, all multiples of 5 and 7 are possible. The possible scores are

5, 7, 10, 12 (5+7), 14, 15, 17 (10+7), 19 (14+5), 20, 21, 22 (15+7), **24** (19+5**), 25, 26** (19+7**),**
27 (20+7**), 28,** (that is five in a row), …

Thus, 23 is the largest score impossible to obtain. (Because 5 and 7 are relatively prime, this result can be obtained by $5 \cdot 7 - 5 - 7 = 23$. See *Frobenius Number*.)

5. The sides of a triangle are 15, 20, and 25. Find the sum of the altitudes.

Solution: (b) This is a right triangle because the sides are in the ratio 3:4:5 or because 25^2 (625) is the sum of $15^2 + 20^2$ (225 + 400).



Hence, the two legs are already altitudes. The remaining altitude h satisfies $20{:}h = 25{:}15$. Thus, $h = 12$ and the sum of the altitudes is $12 + 15 + 20 = 47$.

6. The number 96 can be expressed as the difference of perfect squares $x^2 - y^2$ in four different ways. Find the largest value of such x.

Solution: (a) Using the fact that $x^2 - y^2 = (x - y)(x + y)$, we can factor 96 using $x - y$ and $x + y$. Furthermore, the sum of these factors is $2x$, so the sum of the two factors must be even to produce a difference of squares. We consider the list of all factor pairs of 96:
1 and 96 (sum: 97), **2 and 48 (sum: 50; $x = 25$**) 3 and 32 (sum: 35), 4 and 24 (sum: 28), 6 and 16 (sum: 22), 8 and 12 (sum: 20).

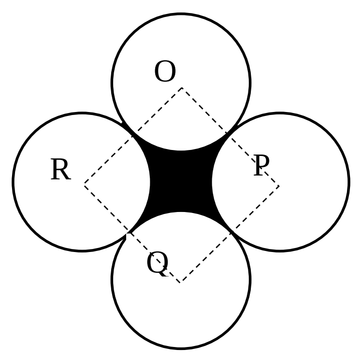
7. An old woman goes to market where a horse steps on her basket and crushes the eggs. The rider offers to pay for the damages and asks her how many eggs she had brought. She does not remember the exact number, but when she had taken them out two at a time, there was one egg left. The same resulted when she picked them out three, four, five, and six at a time, but when she took them seven at a time they came out even. Let n represent the smallest number of eggs she could have had in her basket. Which of the following statements about n is true?

Solution: (e) It is implied that $n - 1$ is divisible by 2, 3, 4, 5, 6 and n itself is divisible by 7. Hence, $n - 1$ is a multiple of 3, 4, and 5 so it is a multiple of 60. We seek number $60k + 1$ that is divisible by 7. The smallest such value is 301.

8. The first ten digits of pi are 3, 1, 4, 1, 5, 9, 2, 6, 5, 3. How many distinct ten digit numbers can be formed with these digits.

Solution: (b) The number of permutations of ten digits is 10! However, due to the duplicate digits 1, 3, and 5, we divide by the permutations of each pair: $\dfrac{10!}{2!\,2!\,2!} = \dfrac{10!}{8}$

9. In the figure, circles O and Q are tangent to circles R and P and each circle has radius 1. What is the area of the shaded region if $\overline{OP} \perp \overline{PQ}$ and $\overline{OP} \perp \overline{OR}$?



Solution: (c) The centers of the circles are vertices of a square with sides of length 2 which intersects each circle in a quarter circle. Thus, the shaded area is $4 - \pi$.

10. Twelve sprinters run in a 100 meter dash. How many ways are possible for the athletes to finish 1st, 2nd, and 3rd?

Solution: (d) There are $12 \cdot 11 \cdot 10 = 1320$ distinct ways for the sprinters to finish.

11. Find the minimum value of the function $g(x) = ex^2 - 2x + \pi$.

Solution: (e) The vertex of this upward opening parabola is (x, y) where $x = -\frac{-2}{2(e)} = \frac{1}{e}$ and the minimum value is $y = g\left(\frac{1}{e}\right) = \pi - \frac{1}{e}$.

12. How many whole numbers lie between $\frac{e}{2}$ and 4π ?

Solution: (d) We note that $1 < \frac{e}{2} < 2$ and $12 < 4\pi < 13$. Therefore, there are $12 - 2 + 1 = 11$ integers between $\frac{e}{2}$ and 4π.

13. In a twelve team tournament, if every team plays every other team twice, how many games are played in the tournament?

Solution: (c) The first team will play 11 games. The second team will play 10 games not including the first team. The third will play 9 games not including the first and second teams, etc. There are a total of $11 + 10 + 9 + \cdots + 2 + 1 = 66$ games with distinct pairs of teams. The tournament will require 132 games.

14. On a twenty-question test, each correct answer is worth 5 points, each unanswered question is worth 1 point and each incorrect answer is worth 0 points. Which of the following scores is NOT possible?

Solution: (e) Getting all 20 questions correct results in a perfect score of 100. Getting 19 correct answers and leaving the other blank (1 pt) results in a score of 96. A score of 97 is not possible.

15. Given $y_{k+2} - y_{k+1} - 2y_k = 0$ for $k = 0, 1, 2, \ldots$, find y_6 if $y_0 = 9$ and $y_1 = -12$.

Solution: (a) Taking a direct approach, we have $y_{k+2} = y_{k+1} + 2y_k$, so

$y_2 = -12 + 2(9) = 6$, $y_3 = 6 + 2(-12) = -18$, $y_4 = -18 + 2(6) = -6$
$y_5 = -6 + 2(-18) = -42$, $y_6 = -42 + 2(-6) = -54$.

16. If $i = \sqrt{-1}$, then $\sum_{k=1}^{183} (-i)^k$ equals

Solution: (b) Enumerating the first several terms of the series, we have

$$\begin{aligned}\sum_{k=1}^{183} (-i)^k &= -i + (-i)^2 + (-i)^3 + (-i)^4 + (-i)^5 + (-i)^6 + (-i)^7 + \cdots + (-i)^{182} + (-i)^{183} \\ &= -i + (-1) + i + 1 + (-i) + (-1) + i + \cdots + (-i)^2 \underbrace{(-i)^{4\cdot 45}}_{=1} + (-i)^3 \underbrace{(-i)^{4\cdot 25}}_{=1} \\ &= (\underbrace{-i - 1 + i + 1}_{0}) + (\underbrace{-i - 1 + i + 1}_{0}) + \cdots + (\underbrace{-i - 1 + i + 1}_{0}) - i - 1 + i \\ &= -1\end{aligned}$$

17. For $f(x)=\dfrac{2x}{1-2x}$, find $f(f(f(x)))$.

Solution: (a) We have

$$f(f(x))=\frac{2\left(\frac{2x}{1-2x}\right)}{1-2\left(\frac{2x}{1-2x}\right)}\cdot\frac{1-2x}{1-2x}$$
$$=\frac{4x}{1-2x-4x}$$
$$=\frac{4x}{1-6x}$$

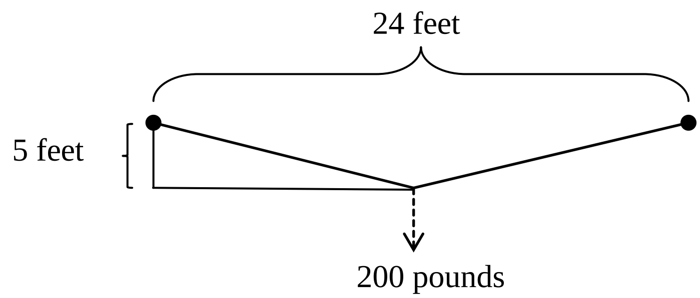
Composing a second time gives

$$f(f(f(x)))=\frac{2\left(\frac{4x}{1-6x}\right)}{1-2\left(\frac{4x}{1-6x}\right)}\cdot\frac{1-6x}{1-6x}$$
$$=\frac{8x}{1-6x-8x}$$
$$=\frac{8x}{1-14x}$$

18. From the set {1, 2, 3, 4, 5, 6, 7, 8, 9}, how many nonempty subsets have elements that sum to an even integer?

Solution: (b) A set of n objects has 2^n subsets (including the empty set). Certainly each subset with only even whole numbers has elements that add to an even integer. There are 2^4 such subsets (including the empty set). We may add any even number of the five odd elements and still have a set whose elements add to an even integer. Hence, there are $2^4\cdot\left[\binom{5}{0}+\binom{5}{2}+\binom{5}{4}\right]=64\cdot[1+10+5]=256$ subsets including the empty set. Therefore, there are 255 nonempty subsets whose elements sum to an even integer.

19. A 200 pound object is hung at the center of a 26 foot cable anchored to a wall at each end as shown in the following diagram. How much force is exerted on each anchor?



Solution: (e) We note that the triangle formed by the force F acting along the cable with the horizontal and vertical forces is proportional to the 5:12:13 right triangle whose hypotenuse is formed by the cable. Each anchor supports exactly half of the 200 pound weight by its vertical component. Therefore, the induced force F along the cable satisfies the proportion

$$\frac{5}{13} = \frac{100}{F}$$

Solving the proportion gives $F = 13 \cdot 20 = 260$ lbs.

20. A circle with center (*h*, *k*) contains the points (–2, 10), (–9, –7), and (8, –14). The circumference of the circle is

Solution: (e) We establish two equations in h and k by equating the distance from the center to two points.

From (–2, 10) to (–9, –7)

$$(h+2)^2 + (k-10)^2 = (h+9)^2 + (k+7)^2$$

From (–2, 10) to (8, –14)

$$(h+2)^2 + (k-10)^2 = (h-8)^2 + (k+14)^2$$

Expanding, the quadratic terms add out leaving

$$14h + 34k = -26 \qquad 20h - 48k = 156$$

from which we obtain $h = 3$ and $k = -2$. The radius of the circle is $\sqrt{(3+2)^2 + (-2-10)^2} = 13$. The circumference is 26π.

21. From a group of 15 mathematics students, 10 were randomly selected to be on a state mathematics team. Let *P* represent the probability that 4 of the 5 top students are included in the selection. Which of the following statements is true?

Solution: (b) There are 5 ways to select 4 students from 5. This is independent of selecting the last 6 team members from the remaining 10. Hence, the probability P is

$$P = \frac{5 \cdot \binom{10}{6}}{\binom{15}{10}} = 5 \cdot \frac{10!}{6!4!} \cdot \frac{10!5!}{15!} = \frac{50}{143}$$

Furthermore, $.2 < P < .4$.

22. The solution set for the inequality $x - \frac{3}{x} > 2$ is given by

Solution: (a) The inequality is equivalent to

$$\frac{(x+1)(x-3)}{x} > 0$$

For this quotient to be positive, we see that x is in the interval $(-1, 0)$ or $(3, \infty)$.

23. Let *A* and *B* be subsets of *U* and denote the complement of subset *X* by X^C. Find

$$\left[[A \cap (A \cap B^C)] \cap B\right]^C$$

Solution: (d) Unraveling the intersection of sets, we have $A \cap (A \cap B^C) = A \cap B^C$ and $(A \cap B^C) \cap B^C = A \cap B^C$. Therefore, the inner bracket of intersections is empty. Its complement is U.

Note: The following are equal: $[A \cap (A \cap B^C] = [A \cap (A \cap B^C)] = A \cap A \cap B^C = A \cap B^C$

24. If |*x*| is large, then $f(x) = \dfrac{x^5 - x^4 + x^3 + x}{x^3 - 1}$ is approximately

Solution: (c) Long division gives

$$\frac{x^5 - x^4 + x^3 + x}{x^3 - 1} = x^2 - x + 1 + \frac{-x^2 + 2x + 1}{x^3 - 1}$$

For large values of $|x|$, the remainder is relatively small so

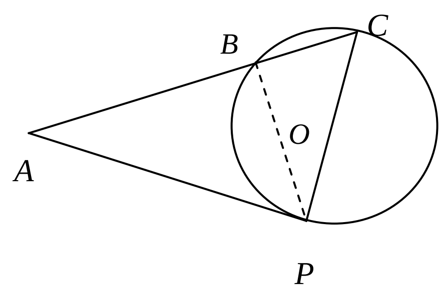
$$\frac{x^5 - x^4 + x^3 + x}{x^3 - 1} \approx x^2 - x + 1$$

25. If $f(x) = 5x^2$, what are all the real values of a and b such that the graph of $g(x) = ax^2 + b$ is below the graph of $y = f(x)$ for all values of x?

Solution: (b) If $a < 5$ then the graph of g is wider than the graph of f. Thus, if b is negative, the vertex of g is below the vertex of f. Therefore, the graph of g is below the graph of f if $a \leq 5$ and b is negative.

26. In the circle shown with center O and radius 2, AP has length 3 and is tangent to the circle at P. If CP is the diameter of the circle, what is the length of BC?

Solution: (d) We note that $\overline{AP} \perp \overline{CP}$ so triangle APC is a right triangle. Therefore, the length of $\overline{AC}$ is 5. Triangle PBC is also a right triangle because segment $\overline{CP}$ is a diameter. Therefore, triangle PBC is similar to triangle APC. Thus, the ratio of $\overline{BC} : \overline{CP}$ is 4:5. Because $\overline{CP} = 4$, solving for $\overline{BC}$ gives $\overline{BC} = 3.2$.



27. Find the coefficient of x^2y^3 in the binomial expansion of $(x - 2y)^5$.

Solution: (c) The Binomial Theorem gives

$$(x-2y)^5 = \sum_{k=0}^{5} \binom{5}{k} x^k (-2y)^{5-k}$$

Therefore, the coefficient we seek is when $k = 2$: $\binom{5}{2}(1)^2(-2)^3 = -80$.

28. Find the equation of the set of all points (x, y) such that the sum of those distances from (0, 1) and (1, 0) is 2.

Solution: (e) We have

$$\sqrt{x^2 + (y-1)^2} + \sqrt{(x-1)^2 + y^2} = 2$$

or

$$\left(\sqrt{x^2 + (y-1)^2}\right)^2 = \left(2 - \sqrt{(x-1)^2 + y^2}\right)^2$$

which produces

$$x^2 + (y-1)^2 = 4 - 4\sqrt{(x-1)^2 + y^2} + (x-1)^2 + y^2$$

Isolating the remaining radical and squaring both sides yields

$$(x - y - 2)^2 = \left(-2\sqrt{(x-1)^2 + y^2}\right)^2$$

Expanding and collecting like terms results in

$$3x^2 + 2xy + 3y^2 - 4x - 4y = 0$$

29. The number of vertices of an ordinary polyhedron with 12 faces and 17 edges is

Solution: (a) Euler's formula $F - E + V = 2$ gives $V = 7$.

30. The inverse of the function $f(x) = \dfrac{x}{x-1}$ **is**

Solution: (c) Interchanging x and y and solving for y gives

$$x = \frac{y}{y-1} \quad \Rightarrow \quad x(y-1) = y \quad \Rightarrow \quad y(x-1) = x$$

so $f^{-1}(x) = \dfrac{x}{x-1}$