Utah State Mathematics Contest
Junior Exam
March 14, 2012

1. What is the units digit of 2012^{2012}?

(a) 2 (b) 0 (c) 6 (d) 4 (e) 8

2. A train, traveling at a constant speed, takes 20 seconds from the time it enters a tunnel that is 300 meters long until it emerges from the tunnel. A bat sleeping on the ceiling of the tunnel is directly above the train for 10 seconds. How long is the train?

(a) 150 m (b) 200 m (c) 250 m (d) 300 m (e) 400 m

3. Three dice are tossed. What is the probability that the numbers shown will all be different?

(a) $\frac{5}{9}$ (b) $\frac{5}{27}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$ (e) $\frac{2}{9}$

4. An archery target has two scoring areas: one worth 5 points and another worth 7 points. What is the largest score impossible to obtain?

(a) 8 (b) 11 (c) 16 (d) 23 (e) 31

5. The sides of a triangle are 15, 20, and 25. Find the sum of the altitudes.

(a) 43 (b) 47 (c) 51 (d) 53 (e) 61

6. The number 96 can be expressed as the difference of perfect squares $x^2 - y^2$ in four different ways. Find the largest value of such x.

(a) 25 (b) 13 (c) 24 (d) 17 (e) 36

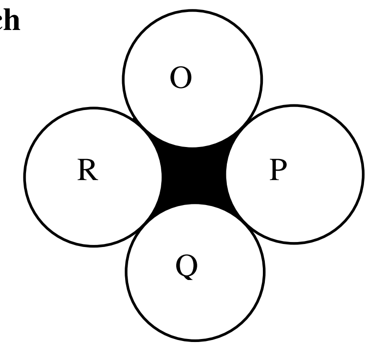
7. An old woman goes to market where a horse steps on her basket and crushes the eggs. The rider offers to pay for the damages and asks her how many eggs she had brought. She does not remember the exact number, but when she had taken them out two at a time, there was one egg left. The same resulted when she picked them out three, four, five, and six at a time, but when she took them seven at a time they came out even. Let *n* represent the smallest number of eggs she could have had in her basket. Which of the following statements about *n* is true?

(a) $2n < 500$ (b) $110 < \frac{n}{4} < 150$ (c) $3 < \frac{n-1}{100} < 4$ (d) $\frac{n}{100} > 7$ (e) $29 < \frac{n-1}{10} < 31$

8. The first ten digits of pi are 3, 1, 4, 1, 5, 9, 2, 6, 5, 3. How many distinct ten digit numbers can be formed with these digits.

(a) 10! (b) $\frac{10!}{8}$ (c) 90 (d) $\frac{10!}{3!}$ (e) 720

9. In the figure, circles O and Q are tangent to circles R and P and each circle has radius 1. What is the area of the shaded region if $\overline{OP} \perp \overline{PQ}$ and $\overline{OP} \perp \overline{OR}$?



(a) $\frac{\pi}{4}$ (b) 1 (c) $4 - \pi$ (d) π (e) 4

10. Twelve sprinters run in a 100 meter dash. How many ways are possible for the athletes to finish 1st, 2nd, and 3rd?

(a) 33 (b) 440 (c) 720 (d) 1320 (e) 1564

11. Find the minimum value of the function $g(x) = ex^2 - 2x + \pi$.

(a) $e + \frac{1}{e}$ (b) $e - \frac{1}{e}$ (c) $\pi + \frac{1}{e}$ (d) $\pi - \frac{2}{e}$ (e) $\pi - \frac{1}{e}$

12. How many whole numbers lie between $\frac{e}{2}$ and 4π ?

(a) 8 (b) 9 (c) 10 (d) 11 (e) 12

13. In a twelve team tournament, if every team plays every other team twice, how many games are played in the tournament?

(a) 66 (b) 80 (c) 132 (d) 144 (e) 196

14. On a twenty-question test, each correct answer is worth 5 points, each unanswered question is worth 1 point and each incorrect answer is worth 0 points. Which of the following scores is NOT possible?

(a) 90 (b) 91 (c) 92 (d) 96 (e) 97

15. Given $y_{k+2} - y_{k+1} - 2y_k = 0$ for $k = 0, 1, 2, \ldots$, find y_6 if $y_0 = 9$ and $y_1 = -12$.

(a) –54 (b) 64 (c) –32 (d) 27 (e) 54

16. If $i = \sqrt{-1}$, then $\sum_{k=1}^{183} (-i)^k$ equals

(a) 0 (b) –1 (c) $-i$ (d) $-1 + i$ (e) $-1 - i$

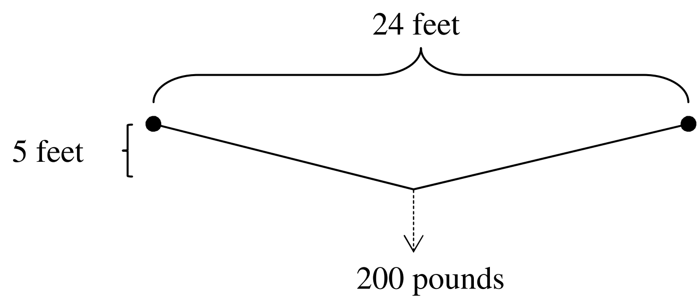
17. For $f(x) = \dfrac{2x}{1-2x}$, find $f(f(f(x)))$.

(a) $\dfrac{8x}{1-14x}$ (b) $\dfrac{8x}{1-16x}$ (c) $\dfrac{2x}{1-8x}$ (d) $\dfrac{2x^3}{1-2x^3}$ (e) $\dfrac{2x}{1-16x}$

18. From the set {1, 2, 3, 4, 5, 6, 7, 8, 9}, how many nonempty subsets have elements that sum to an even integer?

(a) 512 (b) 255 (c) $\frac{9!}{7!\,2!}$ (d) $\frac{9!}{2!}$ (e) 256

19. A 200 pound object is hung at the center of a 26 foot cable anchored to a wall at each end as shown in the following diagram. How much force is exerted on each anchor?



(a) 100 lbs (b) 150 lbs (c) 200 lbs (d) 240 lbs (e) 260 lbs

20. A circle with center (h, k) contains the points (–2, 10), (–9, –7), and (8, –14). The circumference of the circle is

(a) 10π (b) 12π (c) 24π (d) 13π (e) 26π

21. From a group of 15 mathematics students, 10 were randomly selected to be on a state mathematics team. Let *P* represent the probability that 4 of the 5 top students are included in the selection. Which of the following statements is true?

(a) $0 \leq P \leq \frac{1}{5}$ (b) $\frac{1}{5} \leq P \leq \frac{2}{5}$ (c) $\frac{2}{5} \leq P \leq \frac{3}{5}$ (d) $\frac{3}{5} \leq P \leq \frac{4}{5}$ (e) $\frac{4}{5} \leq P \leq 1$

22. The solution set for the inequality $x - \frac{3}{x} > 2$ is given by

(a) $(-1,0) \cup (3,\infty)$ (b) $(-\infty,-1) \cup (0,3)$ (c) $(-\infty,-3) \cup (1,\infty)$

(d) $(-\infty,1) \cup (3,\infty)$ (e) $(-3,0) \cup (1,\infty)$

23. Let *A* and *B* be subsets of *U* and denote the complement of subset *X* by X^C. Find

$$\left[[A \cap (A \cap B^C)] \cap B\right]^C$$

(a) B^C (b) A^C (c) $A \cap B^C$ (d) U (e) $\emptyset$

24. If $|x|$ is large, then $f(x) = \dfrac{x^5 - x^4 + x^3 + x}{x^3 - 1}$ **is approximately**

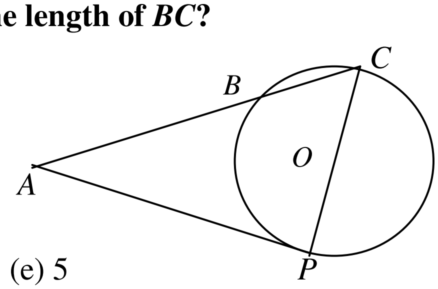
(a) $x^2 + x$ (b) x^2 (c) $x^2 - x + 1$ (d) $x^2 + 1$ (e) $x^2 - x$

25. If $f(x) = 5x^2$, what are all the real values of *a* and *b* such that the graph of $g(x) = ax^2 + b$ is below the graph of $y = f(x)$ for all values of *x*?

(a) $a \geq 5$ and b is positive (b) $a \leq 5$ and b is negative (c) a is negative and b is positive

(d) a is any real number and b is negative (e) $a \geq 5$ and b is negative

26. In the circle shown with center *O* and radius 2, *AP* has length 3 and is tangent to the circle at *P*. If *CP* is the diameter of the circle, what is the length of *BC*?



(a) 1.5 (b) 2.4 (c) 3 (d) 3.2 (e) 5

27. Find the coefficient of x^2y^3 in the binomial expansion of $(x - 2y)^5$.

(a) –160 (b) 80 (c) –80 (d) 8 (e) –8

28. Find the equation of the set of all points (x, y) such that the sum of those distances from (0, 1) and (1, 0) is 2.

(a) $x^2 + xy + y^2 - 2x - 2y + 2 = 0$ (b) $3x^2 - 2xy + 3y^2 - 4x + 4y - 2 = 0$

(c) $4x^2 - 2xy + 4y^2 - 2x - 2y = 0$ (d) $x^2 - 2xy + y^2 + 2x - 2y + 4 = 0$

(e) $3x^2 + 2xy + 3y^2 - 4x - 4y = 0$

29. The number of vertices of an ordinary polyhedron with 12 faces and 17 edges is

(a) 7 (b) 5 (c) 11 (d) 9 (e) 13

30. The inverse of the function $f(x) = \dfrac{x}{x-1}$ **is**

(a) $f^{-1}(x) = \dfrac{x}{x+1}$ (b) $f^{-1}(x) = 1 + \dfrac{1}{x}$ (c) $f^{-1}(x) = \dfrac{x}{x-1}$

(d) $f^{-1}(x) = 1 - \dfrac{1}{x}$ (e) $f^{-1}(x) = \dfrac{x-1}{x+1}$