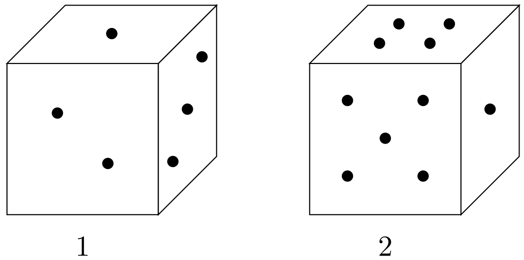
Name:________________________________

State Math Contest – Senior Exam

Instructions:

- Do not turn this page until your proctor tells you.
- Enter your name, grade, and school information following the instructions given by your proctor.
- Calculators are **not** allowed on this exam.
- This is a multiple choice test with 40 questions. Each question is followed by answers marked a), b), c), d), and e). Only one answer is correct.
- Mark your answer to each problem on the bubble sheet Answer Form with a #2 pencil. Erase errors and stray marks. Only answers properly marked on the bubble sheet will be graded.
- **Scoring:** You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
- You will have 2 hours and 30 minutes to finish the test.

1. The following pictures show two views of a non standard die (however the numbers 1 - 6 are represented on the die). How many dots are on the bottom face of figure 2?



a) 1 b) 2 c) 3

d) 4 e) 5

2. Two bicyclists, Annie and Bonnie, are 30 miles apart on a steep road. Annie and Bonnie travel at a constant speed and start riding towards each other at the same time. Annie travels downhill and goes twice as fast as Bonnie. They expect to meet in one hour, but Annie stops for a flat tire after 30 minutes and she is unable to continue. How many minutes should Annie expect to wait for Bonnie if Bonnie continues at the same speed?

a) 45 b) 60 c) 75

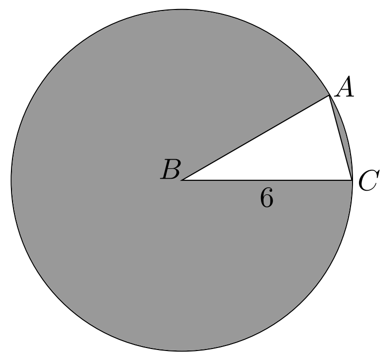
d) 90 e) 105

3. Find the product of all real solutions to the equation $x^4 + 2x^2 - 35 = 0$.

a) 5 b) -5 c) 7

d) -7 e) -35

4. In the following diagram, which is not drawn to scale, the shaded area is 32π. The circle has radius $\overline{BC} = \overline{BA} = 6$. Find the measure of $\angle ABC$.



a) $30°$

b) $40°$

c) $\arcsin\left(\frac{2\pi}{9}\right)$

d) $\arcsin\left(\frac{\sqrt{2}}{3}\right)$

e) $45°$

5. Traveling from Salt Lake City to Denver, a regularly priced adult ticket is discounted by 15% for seniors and 50% for children. Tickets for a party of 3 seniors, 5 adults, and 7 children cost \$884. How much will it cost in dollars for 2 seniors, 6 adults, and 8 children?

a) 884

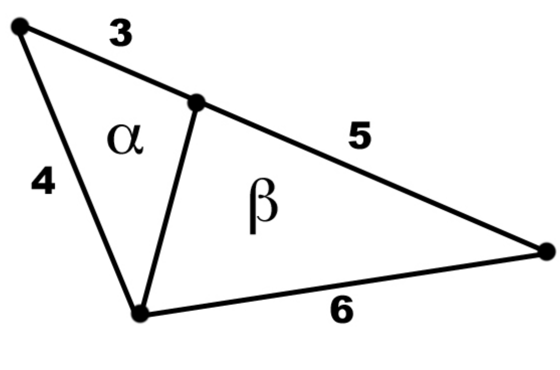
b) 856

c) 868

d) 936

e) 976

6. The large triangle below is divided into two triangles of areas α and β. Find α/β.



a) $\frac{3}{5}$

b) $\frac{1}{2}$

c) $\frac{4}{6}$

d) $\frac{3}{4}$

e) $\frac{7}{11}$

7. Use properties of logarithms to find the exact value of the expression

$$\log_5 2 \cdot \log_2 125.$$

a) 3

b) 2

c) 1

d) 0

e) -1

8. Let

$$f(x) = \begin{cases} 2x & \text{for} \quad 0 \le x \le 0.5 \\ 2(1-x) & \text{for} \quad 0.5 < x \le 1. \end{cases}$$

If $x_0 = \frac{6}{7}$ and $x_n = f(x_{n-1})$ for $n \ge 1$, find x_{100}.

a) $\frac{2}{7}$

b) $\frac{4}{7}$

c) $\frac{6}{7}$

d) $\frac{10}{7}$

e) $\frac{96}{7}$

9. When $3x^{12} - x^3 + 5$ is divided by $x + 1$ the remainder is:

a) 1

b) 3

c) 5

d) 7

e) 9

10. A factor of $243,000,000$ is chosen at random. What is the probability that the factor is a multiple of 9?

a) 0

b) $\frac{1}{6}$

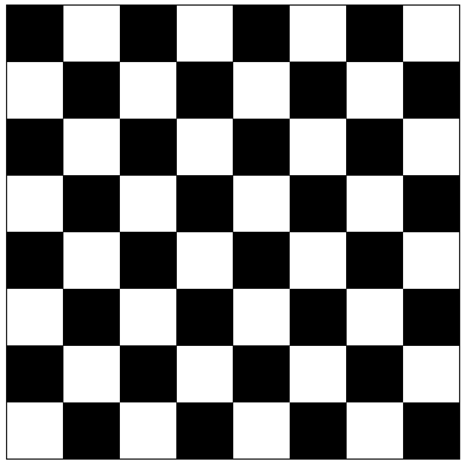
c) $\frac{1}{3}$

d) $\frac{2}{3}$

e) $\frac{1}{3,000}$

11. How many different kinds of pieces can you cut from an 8×8 checkered board consisting of four 1×1 squares that are joined end to end?

Note: Two pieces are the same if one of the pieces can be rotated or translated in the plane to obtain the other piece.



a) 10 b) 12 c) 20

d) 8 e) 16

12. Sally decides to hike to the Y on Y-mountain. The first part of her hike (from her home to the trailhead) was on level ground. On level ground Sally walks at a rate of 3.5 miles per hour. From the trailhead to the Y she hikes at an average rate of 2.4 miles per hour. From the Y back to the trailhead she hikes at about 4.2 miles per hour. Finally, she returns home (once again at a rate of 3.5 miles per hour). Given that two hours have passed between when she leaves home and when she returns and that the total round trip distance is 6.65 miles, how long is the hike from the trailhead to the Y?

a) .6 miles b) .7 miles c) 1.2 miles

d) 1.4 miles e) 2.1 miles

13. How many ways can you write 5 as the sum of one or more positive integers if different orders are not counted differently? For example, there are three ways to write 3 in this way: $1 + 1 + 1$, $1 + 2$, and 3.

a) 7 b) 6 c) 8

d) 5 e) 10

14. How many real solutions does the equation $x^{3/2} - 32x^{1/2} = 0$ have?

a) 0 b) 1 c) 2

d) 3 e) 4

15. When slicing a rectangular cake, what is the smallest number of straight cuts that you need to make exactly 7 pieces?

a) 6 b) 5 c) 7

d) 3 e) 4

16. A very thin disk has an area (on one side) of 4π. A square window is cut into a wall. What is the smallest area (the *lower bound*) the window can have and still be large enough for the disk to fit through?

a) $16/\pi$ b) 4π c) $8\sqrt{2}$

d) 8 e) 16

17. How many whole numbers from 1 to 10000, inclusive, are multiples of 20 but not of 22?

a) 489 b) 478 c) 455

d) 458 e) 432

18. Find the sum of all the fractions strictly between 0 and 1 which, in reduced form, have denominator less than or equal to 10.

a) $\frac{21}{6}$ b) $\frac{43}{4}$ c) 25

d) $\frac{43}{2}$ e) $\frac{31}{2}$.

19. Express as single complex number: $1 + i + i^2 + i^3 + ... + i^{100}$ where $i^2 = -1$

a) i b) -1 c) 1

d) $100i$ e) 0

20. The graph of the function $h(x)$ is a straight line. On the interval $2 \leq x \leq 4$ the function $h(x)$ satisfies $h(x) = 3 + |x-4| + 2|x-6|$. What is $h(7)$?

a) -5 b) -2 c) 0

d) 3 e) 19

21. Let $\epsilon = 10^{-25} = .0000000000000000000000001$, and let $x = \sqrt{1+2\epsilon}$, $y = \sqrt[3]{1+3\epsilon}$ and $z = 1+\epsilon$. Rank the numbers x, y and z. If any of them are equal, say so.

a) $y > x > z$ b) $x < y < z$ c) $x = y = z$

d) $y < x < z$ e) $x = y > z$

22. How many of the following triples can be the side lengths of an obtuse triangle?

$$(2,2,3), (3,5,7), (3,7,11), (7,9,11)$$

a) 0 b) 1 c) 2

d) 3 e) 4

23. Seven students in a classroom are to be divided into two groups of two and one group of three. In how many ways can this be done?

a) 3 b) 35 c) 90

d) 105 e) 315

24. If $|r| < 1$, then $(a)^2 + (ar)^2 + (ar^2)^2 + (ar^3)^2 + \cdots =$

a) $\dfrac{a^2}{(1-r)^2}$ b) $\dfrac{a^2}{1+r^2}$ c) $\dfrac{a^2}{1-r^2}$

d) $\dfrac{4a^2}{1+r^2}$ e) none of these

25. When buying a bike from the *Math Bikes* company, there are three extra options to choose (a bell, a rear fender, and a basket), each of which you can choose to add to the bike or choose not to add it. If *Math Bikes* has sold 300 bikes, what is the largest number of bikes that you can guarantee to have exactly the same extras as each other?

a) 8
b) 37
c) 38
d) 43
e) 292

26. Begin with 63, and keep repeating the following pair of operations: *Add 1, then take the square root.* Thus we generate the following sequence of numbers: $63, 8, 3, 2, \sqrt{3}, \sqrt{1+\sqrt{3}}$, etc. Eventually, those numbers settle down to a *limit*. What is the limit?

a) $1 + \frac{\sqrt{2}}{3}$
b) $\frac{1+\sqrt{5}}{2}$
c) $\sqrt{1+\sqrt{2}}$
d) $\frac{1+\sqrt{2}}{2}$
e) 1

27. A square and an equilateral triangle have the same area. Let A be the area of the circle circumscribed around the square and B be the area of the circle circumscribed around the triangle. Find $\frac{A}{B}$.

a) $\frac{3\sqrt{3}}{8}$
b) $\frac{3\sqrt{3}}{6}$
c) $\frac{3\sqrt{3}}{4}$
d) $\frac{3}{8}$
e) $\frac{3}{4}$

28. Find the number of diagonals that can be drawn in a convex polygon with 200 sides.

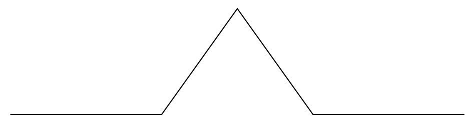
Note: A *diagonal* of a polygon is any line segment between non-adjacent vertices.

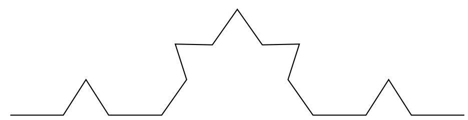
a) 1,969
b) 1,970
c) 20,000
d) 19,700
e) 19,699

29. Koch's curve is created by starting with a line segment of length one. Call this stage 1.

Stage 1

To get from one stage to the next we divide each line segment into thirds and replace the middle third by two line segments of the same length.

Stage 2 

Stage 3 

What is the length of Koch's curve at Stage 6?

a) 16/9 b) 16/27 c) 16/81

d) 1024/81 e) 1024/243

30. For a certain baseball team the probability of winning any game is P, (the probability of winning a particular game is independent of any other games). What is the probability the team wins 3 out of 5 games?

a) $10P^2(1-P)^3$ b) $10P^3(1-P)^2$ c) $5P^3(1-P)^2$

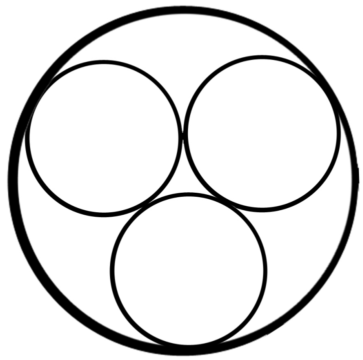
d) $5P^2(1-P)^3$ e) $P^3(1-P)^2$

31. If x is the *fraction* of numbers between 1 and 1,000, inclusive, which contain 4 as a digit, and y is the *fraction* of numbers between 1 and 10,000, inclusive which contain 4 as a digit, what is x/y?

a) 2/3 b) 3/4 c) 27/34

d) 271/3439 e) 2710/3439

32. Given that the area of the outer circle is ten square units, find the area of any one of the three equal circles which are tangent to each other and to the outer circle, and inscribed inside the circle of ten square units.



a) $30(7-4\sqrt{3})$ square units.

b) 2.5 square units.

c) $\dfrac{10}{(3+\sqrt{2})}$ square units.

d) $\sqrt[3]{10}$ square units.

e) 2 square units.

33. Find the product of the zeros of $z^8 + 4z^4 + 16$ that lie in the first quadrant of the complex plane.

a) $\sqrt{3}$

b) $\sqrt{3}i$

c) $1+i$

d) 2

e) $2i$

34. The natives of Wee-jee Islands rate 2 spears as worth 3 fishhooks and a knife, and will give 25 coconuts for 3 spears, 2 knives, and a fishhook together. Assuming each item is worth a whole number of coconuts, how many coconuts will the natives give for each article separately?

a)

Item	Worth in Coconuts
fishhook	1
knife	3
spear	3

b)

Item	Worth in Coconuts
fishhook	1
knife	5
spear	4

c)

Item	Worth in Coconuts
fishhook	2
knife	2
spear	4

d)

Item	Worth in Coconuts
fishhook	2
knife	4
spear	5

e)

Item	Worth in Coconuts
fishhook	3
knife	3
spear	6

35. An octagon in the plane is symmetric about the x-axis, the y-axis, and the line whose equations is $y = x$. If $(1, \sqrt{3})$ is a vertex of the octagon, find its area.

a) $6\sqrt{3}$ b) 11 c) $6 + 2\sqrt{3}$

d) $2 + 6\sqrt{3}$ e) $4 + 4\sqrt{3}$

36. A regular octahedron is formed by setting its vertices at the centers of the faces of the cube. Another regular octahedron is formed around the cube by making the center of each triangle of the octahedron hit at a vertex of the cube. What is the ratio of the volume of the larger octahedron to that of the smaller octahedron?

a) $2\sqrt{2}$ b) $27/8$ c) $3\sqrt{3}$

d) 8 e) 27

37. A square number is an integer number which is the square of another integer.
Positive square numbers satisfy the following properties:

- The units digit of a square number can only be 0, 1, 4, 5, 6, or 9.
- The digital root of a square number can only be 1, 4, 7, or 9.
  The *digital root* is found by adding the digits of the number. If you get more then one digit you add the digits of the new number. Continue this until you get to a single digit. This digit is the digital root.

One of the following numbers is a square. Which one is it?

a) $4,751,006,864,295,101$

b) $3,669,517,136,205,224$

c) $2,512,339,789,576,516$

d) $1,898,732,825,398,318$

e) $5,901,643,220,186,107$

38. Let $f(x) = 9x^2 + dx + 4$. For certain values of d, the equation $f(x) = 0$ has only one solution. For such a value of d, which value of x could be a solution to $f(x) = 0$?

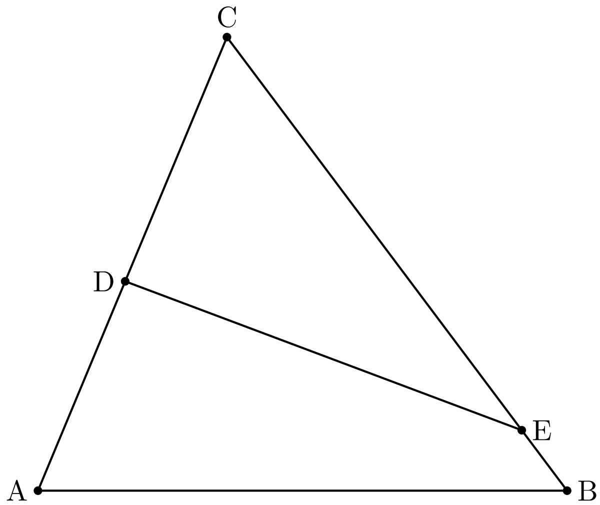
a) $\frac{2}{3}$ b) 1 c) $\frac{4}{3}$

d) 3 e) 12

39. In $\triangle ABC$, $AC = 13$, $BC = 15$ and the area of $\triangle ABC = 84$. If $CD = 7$, $CE = 13$, and the area of $\triangle CDE$ can be represented as $\frac{p}{q}$ where p and q are relatively prime positive integers, find q.

a) 3 b) 5 c) 7

d) 11 e) 13



40. Consider the sequence $1, 9, 5, 7, 6, \frac{13}{2}, \frac{25}{4}, \ldots,$ where each element in the sequence is the *average* of the preceeding two. What is the largest real number smaller than infinitely many elements of the sequence?

a) 7 b) $\frac{19}{3}$ c) $\frac{31}{5}$

d) $\frac{20}{3}$ e) 6

END OF EXAM