Utah State Mathematics Contest
Senior Exam
March 17, 2010

1. In the 2010 Vancouver Olympics, placement in curling semi-finals was based on a 10-team round-robin tournament, in which each of 10 teams plays each other team once, and the top four records move on to the semi-finals. A team is considered to be a strong contender for the gold medal if they win 7 of their 9 matchups. In this system, how many different teams can all end the round-robin tournament with a record of 7 or more wins?

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

2. The midpoints of the three sides of $\triangle ABC$ are $(5, 4)$, $(1, 2)$, and $(-1, 6)$. Which of the following is one of the actual points A, B, or C?

(a) $(4, 9)$ (b) $(3, 8)$ (c) $(5, 2)$ (d) $(6, 3)$ (e) $(0, 6)$

3. What is the greatest number of times that the graph of a 10^{th} degree polynomial can intersect with the graph of a 7^{th} degree polynomial?

(a) 3 (b) 7 (c) 10 (d) 17 (e) 70

4. Havermeyer recently inherited a sizeable sum of money. He paid 30% in taxes and invested 20% of what remained into M&M Enterprises. If M&M received $11,900 from Havermeyer, how much did Havermeyer pay in taxes?

(a) $18,200 (b) $19,600 (c) $17,850 (d) $25,500 (e) $3,570

5. Suppose that the Cartesian three-dimensional space is divided up by the planes $x = y$, $x = z$, and $y = z$. How many sections has the 3-space been divided into?

(a) 4 (b) 5 (c) 6 (d) 8 (e) 12

6. Using each of the digits 1 through 9 once, you can create five numbers, each being one or two digits long. If the five numbers are prime, what is the smallest possible sum of these 5 numbers?

(a) 180 (b) 225 (c) 252 (d) 269 (e) 274

7. Let $|\mathbf{U}-\mathbf{10}|=\mathbf{V},$ given that U < 10. What is the value of U – V?

(a) 10 – 2V (b) -10 (c) 10 – 2U (d) 10 (e) $|\mathbf{U}-\mathbf{10}|-\mathbf{U}$

8. In rectangle PQRS, we have P = (4, 16), Q = (24, 26), and S = (7, y) for some integer y < 16. What is the area of rectangle PQRS?

(a) 120 (b) 130 (c) 135 (d) 140 (e) 150

9. For some real number greater than 1, the difference between its multiplicative inverse and its additive inverse is 4. In which interval does the number lie?

(a) [0, 1) (b) [1, 2) (c) [2,3) (d) [3,4) (e) [4,5)

10. A man leaves for work at the same time each morning. If his travel speed averages 30 miles per hour, he will be 18 minutes late. If his travel speed averages 45 miles per hour, he will arrive 8 minutes early. What average travel speed (in miles per hour) will put him at work precisely on time?

(a) 38 (b) 39 (c) 40 (d) 41 (e) 42

11. A regular octagon, ABCDEFGH, has sides of length 8. What is the area of quadrilateral ABDG?

(a) $\mathbf{96+64\sqrt{2}}$ (b) $\mathbf{64+96\sqrt{2}}$ (c) $\mathbf{32+128\sqrt{2}}$

(d) $\mathbf{128+32\sqrt{2}}$ (e) $\mathbf{80+80\sqrt{2}}$

12. There exists a positive increasing arithmetic sequence A, B, C, D, E such that A, B, E form a geometric sequence. If A = 9, what is D/C?

(a) $\frac{5}{2}$ (b) $\frac{8}{3}$ (c) $\frac{6}{5}$ (d) $\frac{9}{2}$ (e) $\frac{7}{5}$

13. For some integer, n, the quantity (n!) is divisible by the sum (1 + 2 + 3 + ... + n). Which of the following integers is not a possible value of n?

(a) 7 (b) 14 (c) 21 (d) 28 (e) 35

14. Two tour guides are leading seven tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the condition that each guide must take at least one tourist. How many different groupings of guides and tourists are possible?

(a) 118 (b) 120 (c) 122 (d) 124 (e) 126

15. How many ordered pairs of positive integers (c, d), where $c > d$, have the property of having their squares differ by 48?

(a) 3 (b) 4 (c) 6 (d) 8 (e) 12

16. Yarrrrrggghhhh! Literary critics complain that 80% of all fictional pirate captains have an eye-patch, 75% have a hook-hand, 67% have a peg-leg, and 90% have a pet parrot. By this accounting, at least what percent of these pirate captains must have an eye-patch, a hook-hand, a peg-leg *and* a parrot?

(a) 7% (b) 12% (c) 18% (d) 26% (e) 36%

17. A trail going up Mount Olympus maintains at a steady grade of 6%. Over the course of this trail, a traveler would rise in elevation by 270 meters. How much longer would the trail be if the trail had a 4% grade?

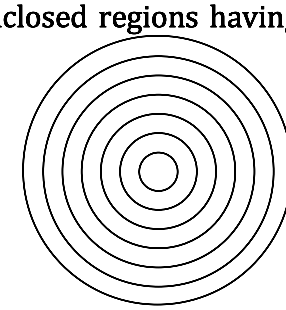
(a) 2250 m (b) 2700 m (c) 3600 m (d) 4500 m (e) 5400 m

18. Suppose that F(x) and G(x) are one-to-one functions. Which of the following is necessarily a one-to-one function?

(a) $(F - G)(x)$ (b) $(F \cdot G)(x)$ (c) $(F / G)(x)$

(d) $(F \circ G)(x)$ (e) $(F + G)(x)$

19. The included picture contains seven concentric circles which have integer radii numbering from 1 to 7. How many non-congruent pairs of enclosed regions having equal area can be found in this picture?



(a) 0 (b) 1 (c) 2

(d) 3 (e) 4

20. A circle passes through the three vertices of an isosceles triangle that has two sides of length 4 and one side of length 2. What is the area of the circle?

(a) $\frac{16\pi}{3}$ (b) $\frac{16\pi}{5}$ (c) $\frac{64\pi}{15}$ (d) $\frac{65\pi}{17}$ (e) $\frac{70\pi}{17}$

21. At one point in US history, it was recommended that roads connect every town to the town nearest to it. Treating each town as a point on a map, and assuming that the distance between any two is distinct (when measured carefully enough), what is the largest number of towns to which any one town may be connected under this system?

(a) 3 (b) 4 (c) 5 (d) 6 (e) 7

22. The ratio $\frac{20^{2009} + 20^{2011}}{20^{2010} + 20^{2010}}$ is closest to which of the following numbers?

(a) 0.05 (b) 0.1 (c) 1 (d) 10 (e) 20

23. In trapezoid ABCD, line segments AB and CD are both perpendicular to segment AD. As well, $AB + CD = BC$, $AB < CD$, and $AD = 9$. What is $AB \cdot CD$?

(a) 20.25 (b) 22.5 (c) 26.25 (d) 28 (e) 29.5

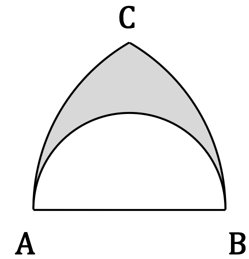
24. Bill randomly selects three different numbers from the set $\{1, 2, 3, 4\}$, while Kerry randomly selects just one number from the set $\{2, 4, 6, 8, 10, 12\}$. What is the probability that Kerry's one number is larger than the sum of Bill's three numbers?

(a) $\frac{3}{8}$ (b) $\frac{1}{3}$ (c) $\frac{5}{12}$ (d) $\frac{5}{8}$ (e) $\frac{2}{3}$

25. A rare book dealer has packaged his books into six bundles; the bundles contain 15, 16, 18, 19, 20, and 31 books. One of the bundles contains only first-editions, while the other five contain re-prints. One customer buys two bundles of re-prints. Another customer also buys only bundles of re-prints, but buys twice as many books as the first customer. The first-edition bundle contains how many books?

(a) 15 (b) 16 (c) 18 (d) 19 (e) 20

26. The following image has the property that A, B, and C are all equidistant from each other. AC is an arc formed from a circle with center B and radius AB, and BC is an arc formed from a circle with center A and radius AB. The white portion of the picture is a semi-circle with diameter AB. Find the ratio of the area of the shaded region to the area of ABC.



(a) $\frac{4\pi}{6\pi-4\sqrt{3}}$ (b) $\frac{3\pi-3\sqrt{3}}{6\pi-4\sqrt{3}}$ (c) $\frac{\pi}{4\pi-2\sqrt{3}}$

(d) $\frac{2\pi}{12\pi-9\sqrt{3}}$ (e) $\frac{5\pi-6\sqrt{3}}{8\pi-6\sqrt{3}}$

27. The repeating decimal of form of a number is $0.0\overline{378}$... If such a number were converted to a fraction and reduced to lowest form, what would be the denominator?

(a) 185 (b) 333 (c) 370 (d) 1111 (e) 9990

28. The positive integers C, D, C + D, C – D are all prime. The sum of these four primes must be divisible by which of the following?

(a) 2 (b) 3 (c) 5 (d) 7 (e) The sum is prime

29. Evan has the same birthday as his grandson, James. For the last six consecutive years, Evan's age was a multiple of James'. How old will Evan be at the next time that Evan's age will again be a multiple of his grandson's?

(a) 54 (b) 64 (c) 70 (d) 72 (e) 80

30. Let $\mathbf{h}(\mathbf{t}) = \frac{16t^4+250t}{4t^2-25}$ for $\mathbf{t} \neq -\frac{5}{2}$. For $\mathbf{h}(\mathbf{t})$ to be continuous, what value must be assigned to $\mathbf{h}(\mathbf{t})$ when $\mathbf{t} = -\frac{5}{2}$?

(a) $\frac{125}{2}$ (b) $\frac{75}{2}$ (c) $\frac{75}{4}$ (d) $\frac{125}{8}$ (e) $\frac{75}{8}$

31. What is the probability that a randomly selected 10-digit number contains all ten different digits?

(a) $\frac{9\cdot 9!}{10^{10}}$ (b) $\frac{10!-9!}{10^9-1}$ (c) $\frac{9!}{9\cdot 10^9}$ (d) $\frac{10!-9!}{10^{10}-1}$ (e) $\frac{9!}{10^9}$

32. A bag contains two gems, either of which may be a ruby or an emerald (with equal probability). An emerald is added to the bag, the contents are well-shaken, and a random stone is drawn from the bag. If the stone drawn out is an emerald, what is the probability that the bag contains a ruby?

(a) $\frac{5}{7}$ (b) $\frac{7}{8}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$ (e) $\frac{5}{8}$

33. Suppose that $30^A = 6$, $30^B = 10$, and $30^C = 15$. What is the mean of A, B, and C?

(a) $\frac{4}{9}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{2}{9}$ (e) $\frac{1}{3}$

34. Assuming that A, B are positive integers, which of the following could be the total of the multiplicities of all negative real zeros for the following polynomial?

$$P(x) = x^9 + 4x^8 - 3x^7 - 26x^6 + 3x^5 + 108x^4 + 107x^3 - 46x^2 - Ax - B$$

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

35. One of the two real roots of the quadratic equation, $ax^2 + bx + c = 0$, is three times the other. Which of the following expressions is equal to the discriminant of the quadratic function?

(a) $\frac{4ac-b^2}{2a}$ (b) $\frac{4ac}{3b^2}$ (c) $\frac{3ac}{b^2}$ (d) $\frac{3ac}{2}$ (e) $\frac{4ac}{3}$

36. What portion of all positive integers are not multiples of 2, 3, 4, 5 or 6?

(a) $\frac{7}{30}$ (b) $\frac{4}{15}$ (c) $\frac{3}{10}$ (d) $\frac{1}{3}$ (e) $\frac{11}{30}$

37. Over the interval [0, 4], how many times does the function $f(x) = \frac{1}{4}\cos(4x)$ reach a local minimum?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

38. Consider the following parametric equations:

$$x = 20\sin t + 5 \qquad y = -10\cos t - 4$$

Which of the following best describes the image made by the graph of these equations in the x-y plane?

(a) A circle (b) An ellipse (c) A parabola

(d) A hyperbola (e) A cone

39. The 3rd-degree polynomial function $f(x) = ax^3 + bx^2 + cx + d$ passes through the points $(-2, 0)$, $(2, 0)$, and $(0, 4)$. What is the value of b?

(a) -2 (b) -1 (c) 0 (d) 1 (e) 2

40. Once a month, Angus and Maggie hike from their home to a nearby mountain; they proceed up to the top, take the same path back down again, and back home. On the road to and from the mountain, their pace is 4 miles per hour. Going up the mountain, they travel 3 miles per hour, while their speed increases to 6 miles per hour on the trip down. If the whole trip takes 3.5 hours, how many miles is the round-trip?

(a) 14 (b) 15 (c) 16 (d) 18 (e) Not enough information