Suggested Talks and Problems

Talk Topics for 2022-2023 School Year

Below are some possible topics. I hope these include something for each taste. (I can also make other suggestions.)

Mathematics Education

1. Geometric Solutions of Cubic Equations - The 11th century Persian poet, mathematician, and astronomer Omar Khayyam devised a geometrical method to also solve the cubic equation $ax^3 + bx^2 + cx + d = 0$. There are many references for this problem.
2. Elementary Methods for Solving Calculus Problems - Many calculus problems can be solved using algebra and geometry, without using the mathematical tools developed in calculus. The Armenian astrophysicist Mamikon A. Mnatsakanian went to Caltech after the fall of Soviet Union and returned to his first love of developing geometric and visual methods for solving calculus problems. There he worked with the famous American mathematician Tom M. Apostol at "Project MATHEMATICS!". Any of their papers include a wealth of insight and are great for presentations at any level. The first paper was titled "A Visual Approach to Calculus Problems" which you can find below. References: http://calteches.library.caltech.edu/4007/1/Calculus.pdf, http://www.its.caltech.edu/~mamikon/calculus.html and http://www.projectmathematics.com/.
3. Short biographical talks on female mathematicians, for example, Margherita Piazzolla Beloch, Elizabeth Smith, etc. See https://en.wikipedia.org/wiki/List_of_women_in_mathematics.
4. Present the paper *An Alternative to Integration by Partial Fractions Technique*. This is the title of a short paper by Yusuf Gurtas published in The College Mathematics Journal, vol. 50 (2019), no. 2, 140-142, and available at https://www.tandfonline.com/doi/full/10.1080/07468342.2019.1561125.
5. Present the paper *A Simple Proof of Descartes' Rule of Sign*. This is the title of a short paper by Xiaoshen Wang published in Amer. Math. Monthly, vol. 111 (2004), no. 4, 525-526, and available at https://www.tandfonline.com/doi/abs/10.1080/00029890.2004.11920108.

Applied Mathematics

6. Presentation of the calculus based proof of $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}$. You learned in Calculus II that this infinite series is convergent (p-series with $p = 2 > 1$). In Boundary Value Problems, you use Fourier series to find its value. But this can be done elegantly using calculus and is included in the book *Proofs from THE BOOK* by Martin Aigner and Günte Ziegler, available in the library. However, there are other proofs. The paper http://math.cmu.edu/~bwsulliv/basel-problem.pdf lists several solutions with references.

7. With COVID-19 epidemic, the interest in mathematical models of diseases is on the rise. They span statistical, discrete, and continuous models. For example, discrete or continuous SIR or SEIR models. Any overview talk of such models or specific development of models for COVID-19 will make an interesting talk.
8. Fibonacci Day is Nov 23 (1123). This year a talk on difference equations, which leads to Fibonacci sequence formula, will be very informative. A good source is the textbook An Introduction to Difference Equations, by Saber Elaydi, ISBN: 978-0387230597, available in the library.

Mathematics

9. Almost Perfect Numbers - For a natural number n, call sum of its proper divisors s(n). A natural number is perfect if it is sum of its proper divisors: s(n)=n. For example, 6 is a perfect number since 1+2+3=6. A natural number is called almost perfect (also called perfect-minus-one) if sum of its proper divisors is one less than the number: s(n)=n-1. For example, 8 is an almost perfect number since 1+2+4=8-1. The only known almost perfect numbers are of powers of 2. The only known odd almost perfect number is $2^0 = 1$. The research project is to quantify, as much as possible, almost perfect numbers. This investigation can include both theoretical and computational work. References: http://www.jstor.org/stable/2689036, http://www.jstor.org/stable/2303162 and http://www.ams.org/journals/mcom/1981-36-154/S0025-5718-1981-0606516-3/.
10. Pi Day (Mar 14, 3/14). A talk on number π; history, calculation, properties, and applications. Last year we had a talk on its irrationality, a talk on pi being transcendental will complete proof of these important properties.
11. Proof of Convergence of Fourier Series. Read, understand, and rewrite the proof in your own words. Reference: David Powers, BVP's and PDE's, 6th edition, ISBN 978-0-12-374719-8.
12. Present the paper *Finding Real Roots of Polynomials Using Sturm Sequences*. This article compares Descartes' Rule of Signs, the Budan-Fourier Theorem, and versions of Sturm's Method in contrast with the approximate root count gleaned from graphing utilities. This articles is published at PRIMUS, (30)1:36-49, 2020, and also available at https://www.tandfonline.com/doi/full/10.1080/10511970.2018.1501626.
13. Present the paper *Inflating the Cube Without Stretching*. This is the title of a short paper by Igor Pak published in Amer. Math. Monthly, vol. 115 (2008), no. 5, 443-445, and available at http://www.math.ucla.edu/~pak/papers/milka2.pdf. In this article the author describes a way to deform a cube that distances between points are maintained while the volume is increased. The beauty of this article is that one can actually construct the deformed cube!
14. Address the conjectures in the paper https://maa.tandfonline.com/doi/pdf/10.1080/0025570X.2020.1704613 regarding certain primes. One such prime is 29 for which $2(29) + 9 = 67$ is also a prime and again $6(67) + 7 = 409$ is another prime.

15. The Arithmetic Mean – Geometric Mean (AM-GM) Inequality has many proofs and applications. A new proof is at https://maa.tandfonline.com/doi/pdf/10.1080/07468342.2020.1697605. An overview of this topic and presentation of interesting proofs will make a nice talk.

Statistics/Data Science

16. Give a talk on probability of randomly selected positive integers being relatively prime. A quick search will result in many references.
17. Present the paper Absent-Minded Passengers published in American Mathematical Monthly, 126:10,867-875, and available at https://www.tandfonline.com/doi/full/10.1080/00029890.2019.1656024. Here is the abstract of the paper. Passengers board a fully booked airplane in order. The first passenger picks one of the seats at random. Each subsequent passenger takes his or her assigned seat if available, otherwise takes one of the remaining seats at random. It is well known that the last passenger obtains her own seat with probability 1/2. We study the distribution of the number of incorrectly seated passengers, and we also discuss the case of several absent-minded passengers.
18. Present the paper Three Persons, Two Cuts: A New Cake-Cutting Algorithm in Mathematics Magazine, Vol 95, Issue 2, 110-122, and available at https://maa.tandfonline.com/doi/full/10.1080/0025570X.2022.2023300

Problems for Fall 2022

PROBLEMS

Proposals

To be considered for publication, solutions should be received by November 1, 2022.

2146. *Proposed by Kenneth Fogarty, Bronx Community College (emeritus), Bronx, NY.*

Let a and d be integers with $d > 0$. We say that (a, d) is *good* if there is an arithmetic sequence with initial term a and difference d that can be split into two sequences of consecutive terms with the same sum. In other words, there exist integers k and n with $0 < k < n$ such that

$$\sum_{i=0}^{k-1}(a+di) = \sum_{i=k}^{n-1}(a+di)\,.$$

If there is no such arithmetic sequence, we say that (a, d) is *bad*.

(a) Show that if $2a > d$, then (a, d) is good.
(b) Show that if $2a = d$, then (a, d) is bad.
(c) Show that if $a = 0$ (and hence $2a < d$), then (a, d) is good.
(d) Show that if $2a < d$ and $a \neq 0$, then there is a d such that (a, d) is good and a d such that (a, d) is bad.

2147. *Proposed by Lokman Gökçe, Istanbul, Turkey.*

Evaluate

$$\prod_{n=2}^{\infty}\frac{n^4+4}{n^4-1}.$$

Math. Mag. **95** (2022) 242–250. doi:10.1080/0025570X.2022.2061246 © Mathematical Association of America

We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

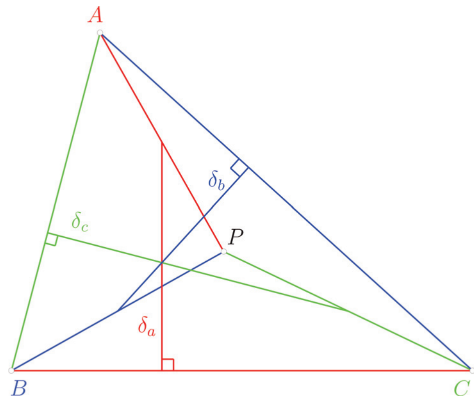
Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at http://mathematicsmagazine.submittable.com. *More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by LaTeX source. General inquiries to the editors should be sent to* mathmagproblems@maa.org.

2148. *Proposed by Tran Quang Hung, Hanoi, Vietnam.*

Let P be an interior point of triangle ABC. Denote by δ_a, δ_b, and δ_c the distances from midpoints of segments PA, PB, and PC to the lines BC, CA, and AB. Prove that

$$PA + PB + PC \geq \delta_a + \delta_b + \delta_c.$$

Show that equality holds if and only if triangle ABC is equilateral and P is its center.



2149. *Proposed by Ioan Băetu, Botoşani, Romania.*

Let $a_1, a_2, \ldots$ be a sequence of integers greater than 1. The series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\prod_{i=1}^{k} a_i} = 1 - \frac{1}{a_1} + \frac{1}{a_1 a_2} - \frac{1}{a_1 a_2 a_3} + \cdots$$

converges by the alternating series test.

(a) If the sequence $a_1, a_2, \ldots$ is unbounded, show that the sum of the series is irrational.

(b) Give an example of a bounded sequence of a_i's such that the sum of the series is irrational.

2150. *Proposed by Matthew McMullen, Otterbein University, Westerville, OH.*

Find the maximum area of a triangle whose vertices lie on the cardioid $r = 1 + \cos\theta$.

PROBLEMS AND SOLUTIONS

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This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively *both* by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

Proposed problems should be sent to **Greg Oman**, either by email (preferred) as a pdf, TEX, or Word attachment or by mail to the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problem is not under consideration by another journal.

Solutions to the problems in this issue should be sent to **Chip Curtis**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above, no later than November 15, 2022. Sending both pdf and TEXfiles is ideal.

PROBLEMS

1226. *Proposed by George Apostolopoulos, Messolongi, Greece.*

Let a, b, and c be positive real numbers. Prove that $\ln \frac{27abc}{(a+b+c)^3} \leq \frac{(a-b)^2+(b-c)^2+(c-a)^2}{3}$.

1227. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

Do there exist functions $f\colon (0,1) \to \mathbb{R}$ and $g\colon (0,1) \to \mathbb{R}$ such that for all x and $y \in (0,1)$, the following two conditions are satisfied:

1. $f(x) < g(x)$, and
2. if $x < y$, then $g(x) < f(y)$?

Either find examples of such f and g or prove that no such f and g exist.

1228. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let R be a ring, and let $f\colon R \to R$ be a function. Say that f is *multiplicative* if $f(xy) = f(x)f(y)$, $f(0) = 0$, and (if R has an identity) $f(1) = 1$. Find all commutative rings R (not assumed to have an identity) with the following two properties:

1. There exists an element $a \in R$ which is not nilpotent, and
2. every multiplicative map $f : R \to R$ is either the identity map or the zero map.

doi.org/10.1080/07468342.2022.2040871

1229. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ii} = 0$ and $a_{ij} = b_i c_j$ for $i \neq j$, where $b_i > 0$ and $c_j \geq 0$ for $1 \leq i, j \leq n$. Prove that the spectral radius of A is strictly less than 1 if and only if $\sum_{i=1}^{n} \frac{b_i c_i}{b_i c_i + 1} < 1$.

1230. *Proposed by Jason Zimba, Amplify, New York, NY.*

A Heronian triangle is a triangle with positive integer side lengths and positive integer area. Denoting the side lengths of a Heronian triangle by a, b, and c, the triangle is called *primitive* if $\gcd(a, b, c) = 1$. We shall say that a primitive Heronian triangle *has an equivalent rectangle* if there exists a rectangle with integer length and width that shares the same perimeter and area as the triangle. Show that infinitely many primitive Heronian triangles have equivalent rectangles.

PI MU EPSILON: PROBLEMS AND SOLUTIONS: SPRING 2022

STEVEN J. MILLER (EDITOR)

1. Problems: Spring 2022

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published (if the solution is not in LaTeX, we are happy to work with you to convert your work). Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged.

#1382: *Proposed by George Stoica, Saint John, New Brunswick, Canada.* Find all integers a, b, c, d such that $a^3 + b^3 + c^3 + d^3 = 2021$; note the integers may be zero or negative. Due to a backlog this problem is appearing in 2022 instead of 2021; if you wish, consider instead $1^3 + a^3 + b^3 + c^3 + d^3 = 2022$.

#1383: *Proposed by Kenneth Davenport.* Consider the following collection of numbers:

$$(1), \quad \begin{pmatrix} 1 & 3 \\ 5 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 7 \\ 5 & 9 & 15 \\ 11 & 17 & 25 \end{pmatrix}, \dots$$

that is to say in the $n \times n$ matrix we have a_{ij}, the entry in the i^{th} row and j^{th} column, equals $j^2 + (2i-3)j + (i^2 - i + 1)$.
(a) For what values of n is the sum of the elements of the $n \times n$ matrix a square? For example, if $n = 25$ the sum is $18255 = 135^2$.
(b) Consider the sum of the elements on the non-main diagonal:

$$\begin{aligned} 1 &= 1^3 \\ 3+5 &= 2^3 \\ 7+9+11 &= 3^3 \\ 13+15+17+19 &= 4^3. \end{aligned}$$

Does this pattern continue? If yes prove it; if not, why not (finding a counter-example suffices).

#1384: *Proposed by Steven J. Miller (Williams College).* Dirichlet's Theorem of Primes in Arithmetic Progression states that if a and b are relatively prime, then there are infinitely many primes congruent to b modulo a (equivalently, there are infinitely many n such that $an + b$ is prime). Prove that if this is all we know about primes, it cannot be enough to prove the Twin Prime Conjecture (there are infinitely many primes p such that $p + 2$ is also prime). One way to do this is to construct a sequence of integers $\{q_i\}_{i=1}^{\infty}$ with $q_{i+1} > q_i + 2$ such that for any relatively prime a and b (other than $a = 2$ and $b = 1$) we have infinitely many n such that $an + b = q_i$ for some i.

#1385: *Proposed by Hongwei Chen, Christopher Newport University, Newport News, Virginia.* A continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

where the a_i are positive integers for all $i \in \mathbb{N}$. It is often denoted by $[a_0; a_1, a_2, \ldots]$.

Continued fractions arise naturally in long division and in the theory of approximation to real numbers by rationals. It is known that every finite continued fraction represents a rational number and every periodic continued fraction represents an irrational root of a quadratic equation with integral coefficients. A famous example is $[1, 1, 1, \ldots] = [\,\overline{1}\,] = \phi$, the golden ratio.

This problem is inspired by the recent CMJ Problem 1186 (51:5, 386 (2020)): *Find a closed-form expression for the continued fraction* $[1, 1, \ldots, 1, 3, 1, 1, \ldots, 1]$*, which has* n *ones before and after, the middle three.*

We found the required closed-form expression is

$$x_n := \frac{F_{n+4}F_{n+1}}{F_{n+2}^2},$$

where F_n is the nth Fibonacci number.

#1386: *Proposed by Gerard Dion, in memory of Zachary Dion.* An interesting construction of Euclidean Geometry is the tangent to two circles with a straightedge and compass. There are 4 possible tangents that are not redundant: 2 for the non-overlapping case (internal and external tangents), 1 for the overlapping case, and 1 for the touching circles case. In the standard construction for non-overlapping circles and external tangent, a new circle is constructed that is inside the larger one, having a radius that is the difference of the other two; see Figure 2.

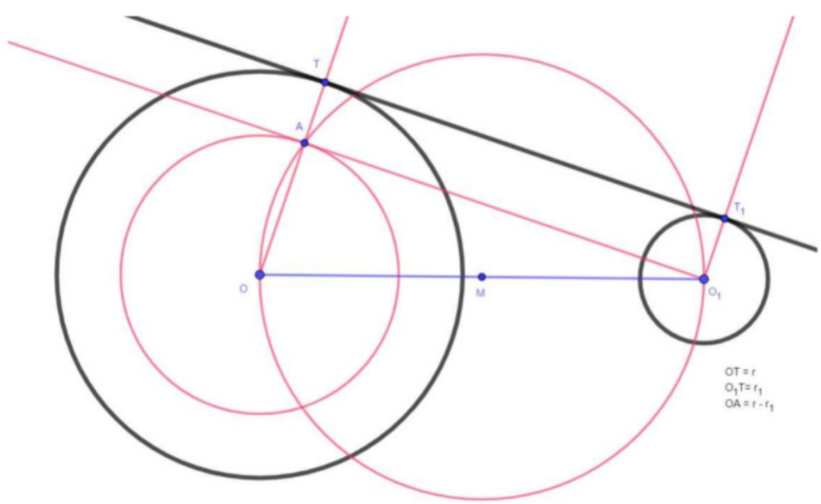


Figure 2. Construction of tangent to two circles.

Prove the constructions for both the overlapping and non-overlapping cases, presented in Figure 3, also work. One novelty of these is that they remain valid when the construction is done in a tool like Geogebra and the circles are resized to switch which circle is largest.

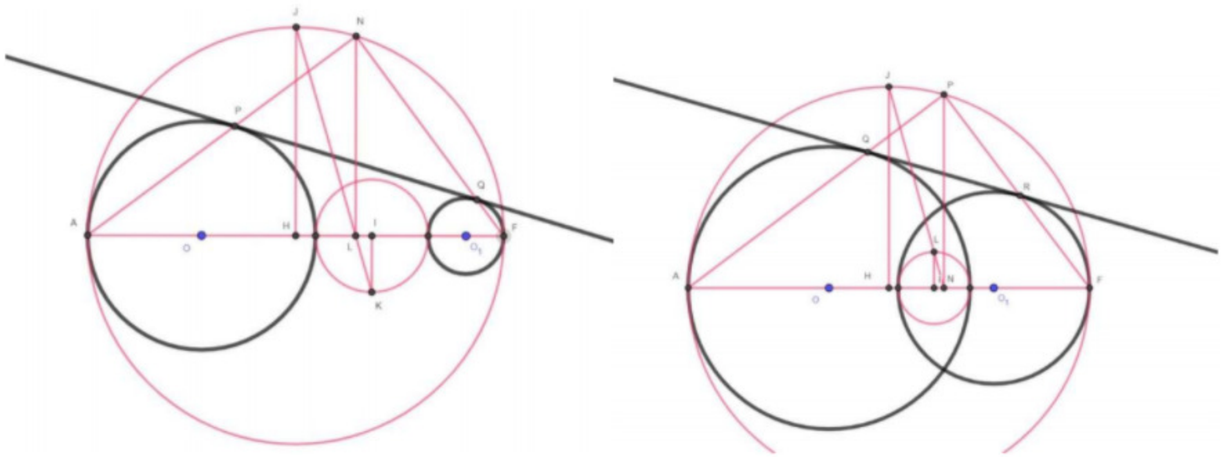


FIGURE 3. Construction of tangent to two circles.

#1387: *Proposed by Hongbiao Zeng, Fort Hays State University, Hays, KS.* Show that for any positive integer n, if p is a prime number such that $n < p < 2n$, then $\binom{2n}{p} \equiv 1 \pmod{p}$.

THE PLAYGROUND

Welcome to the Playground! Playground rules are posted on page 33, except for the most important one: *Have fun!*

THE SANDBOX

In this section, we highlight problems that anyone can play with, regardless of mathematical background. But just because these problems are easy to approach doesn't mean that they are easy to solve!

Limitations of a Limit Theorem (P438). George Stoica (Saint John, Canada) proposed this problem. Let a_i be a sequence of real numbers. A standard calculus theorem states that if $\lim_{n\to\infty}\sum_{i=1}^{n} a_i$ converges then $\lim_{i\to\infty} a_i = 0$. Prove that the conclusion $\lim_{i\to\infty} a_i = 0$ can be false under the weaker hypothesis that $\lim_{n\to\infty}\sum_{i=\lfloor n/2\rfloor+1}^{n} a_i$ exists.

THE MONKEY BARS

These open-ended problems don't have a previously-known exact solution, so we intend for readers to fool around with them. The Playground will publish the best submissions received (proofs encouraged but not required).

Consecutively and Squarely Correct (P439). George Berzsenyi (Rose-Hulman Institute of Technology and the first editor of this column) suggests the next problem, which is attributed to Bernard Recamán (Bogotá, Colombia). A positive integer is *squarely correct* if it is a perfect square or if its base-10 representation consists entirely of adjacent blocks of digits that are positive perfect squares. For example, 99 and 100 are two consecutive numbers that are both squarely correct. However, 101 is not squarely correct—all-zero blocks are not allowed.

1) Are there infinitely many pairs of consecutive squarely correct numbers?
2) Is it possible to find three or more consecutive squarely correct numbers?

THE ZIP LINE

This section offers problems with connections to articles that appear in the magazine. Not all Zip Line problems require you to read the corresponding article, but doing so can never hurt, of course.

***e*-rrational Multiplication (P440).** This problem from Christopher Havens (Twin Rivers PMP) connects to his article with Amy Shell-Gellasch "Fibonacci Meets the Pharaohs: The Decomposition of Multiplication" (p. 24). Christopher asks us to consider a multiplication based on Euler's constant e.

Let s be a real number and

$$s = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

be its continued fraction expansion where the nth convergent of s is given by $p_n / q_n = [a_0; a_1, a_2, \ldots, a_n]$. The *s-Ostrowski decomposition* of a natural number m consists of an integer t and a unique sequence $(c_k)_{k=0}^{t}$ with

$$m = \sum_{k=0}^{t} c_k q_k = c_0 q_0 + c_1 q_1 + \cdots + c_t q_t$$

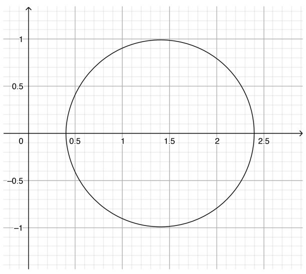
such that $0 \le c_k \le a_{k+1}$ for all k, $0 \le c_0 < a_1$, and for all $k > 0$, if $c_k = a_{k+1}$ then $c_{k-1} = 0$.

Let $s = e = [2; \overline{1, 2k, 1}]_{k=1}^{\infty}$. Find the e-Ostrowski decomposition of 179 and use it to compute the product of 179 and 42 following the suggestion from the article that Fibonacci multiplication can be adapted to other recursively defined sequences.

THE JUNGLE GYM

Any type of problem may appear in the Jungle Gym—climb on!

Figure 1. A circular graph?



Circle Sleuth (P441). Gregory Dresden (Washington & Lee University) plotted the graph of the polar curve $r=\sqrt{\cos 2\theta}+(7/5)\cos\theta$ in the Cartesian plane for values of θ where $\sqrt{\cos 2\theta}$ exists, shown in figure 1. Determine (with justification) whether the graph is a circle.

SUBMISSION AND CONTACT INFORMATION

The Playground features problems for students at the undergraduate and (challenging) high school levels. Problems and solutions should be submitted to *MHproblems@maa.org* and *MHsolutions@maa.org*, respectively (PDF format preferred). Paper submissions can be sent to Jeremiah Bartz, UND Math Dept., Witmer Hall 313, 101 Cornell St. Stop 8376, Grand Forks, ND 58202-8376. Please include your name, email address, and affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.

The deadline for submitting solutions to problems in this issue is October 31, 2022.

PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, and Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

Proposed problems, solutions, and classics should be submitted online at americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below must be submitted by October 31, 2022. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

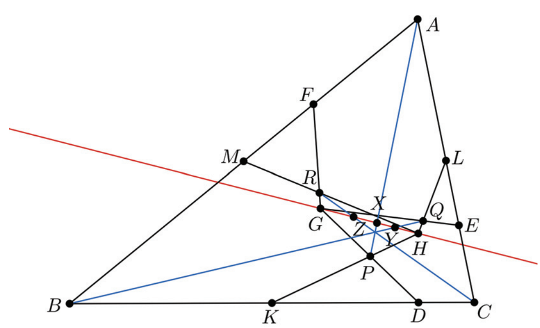
PROBLEMS

12328. *Proposed by Peter Koymans and Jeffrey Lagarias, University of Michigan, Ann Arbor, MI.* An integer binary quadratic form is a function $f : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $f(m,n) = am^2 + bmn + cn^2$ for some $a, b, c \in \mathbb{Z}$. The *value set V(f)* of such a form is defined to be $\left\{ f(m,n) : (m,n) \in \mathbb{Z}^2 \right\}$.
(a) Prove that if $f_1(m,n) = m^2 - mn - 3n^2$ and $f_2(m,n) = m^2 - 13n^2$, then $V(f_1) = V(f_2)$.
(b) Prove that if $f_1(m,n) = m^2 - mn - 4n^2$ and $f_2(m,n) = m^2 - 17n^2$, then $V(f_2) \subseteq V(f_1)$ but $V(f_1) \neq V(f_2)$.

12329. *Proposed by Leonard Giugiuc, Drobeta-Turnu Severin, Romania.* Let n be a positive integer with $n \geq 3$. For each positive integer m with $m \geq 2$, find all real values λ_m such that there are m distinct unit vectors $v_1, \dots, v_m$ in $\mathbb{R}^n$ satisfying $v_i \cdot v_j = \lambda_m$ for all i, j with $1 \leq i < j \leq m$.

12330. *Proposed by Oleh Faynshteyn, Leipzig, Germany.* In the acute and scalene triangle ABC, let G be the centroid, H be the orthocenter, D, E, and F be the feet of the altitudes from A, B, and C, respectively, and K, L, and M be the midpoints of BC, CA, and AB, respectively. Let P be the intersection of DG and KH, let Q be the intersection of EG and LH, and let R be the intersection of FG and MH.
(a) Prove that AP, BQ, and CR are concurrent.
(b) Let X, Y, and Z be the points where GH intersects AP, BQ, and CR. Prove

$$\frac{HX}{XG} + \frac{HY}{YG} + \frac{HZ}{ZG} = 3.$$



http://dx.doi.org/doi.org/10.1080/00029890.2022.2051930

12331. *Proposed by WeChat Group on Matrix Analysis, Nova Southeastern University, Fort Lauderdale, FL.* Let A and B be complex m-by-n matrices, and let C be a complex n-by-m matrix. Prove that if there are nonzero scalars x and y such that $ACB = xA + yB$, then $ACB = BCA$.

12332. *Proposed by Finbarr Holland, University College, Cork, Ireland.* Prove

$$\int_0^\infty \frac{\tanh^2 x}{x^2}\,dx = \frac{14\,\zeta(3)}{\pi^2},$$

where $\zeta(3)$ is Apéry's constant $\sum_{k=1}^{\infty} 1/k^3$.

12333. *Proposed by Moshe Rosenfeld, University of Washington, Seattle, WA, and Tacoma Institute of Technology, Tacoma, WA.* Let G be the multigraph obtained by replacing each edge of the complete graph K_{12} by five edges. Show that the 330 edges of G can be partitioned into 11 sets such that each set forms a graph isomorphic to the icosahedron.

12334. *Proposed by Florin Stanescu, Şerban Cioculescu School, Găeşti, Romania.* Let f be a real-valued function on $[0, 1]$ with a continuous second derivative. Assume that $f(0) = 0$, $f'(0) = 1$, $f''(0) \neq 0$, and $0 < f'(x) < 1$ for all $x \in (0, 1]$. Let $x_1, x_2, \ldots$ be a sequence with $0 < x_1 \leq 1$ and with

$$x_n = f\left(\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right)$$

for $n \geq 2$. Prove $\lim_{n\to\infty} x_n \ln n = -2/f''(0)$.

PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, and Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

Proposed problems, solutions, and classics should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below must be submitted by December 31, 2022. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12335. *Proposed by Tom Karzes, Sunnyvale, CA, Stephen Lucas, James Madison University, Harrisonburg, VA, and James Propp, University of Massachusetts, Lowell, MA.* A *Gaussian integer* is a complex number z such that $z = a + bi$ for integers a and b. Show that every Gaussian integer can be written in at most one way as a sum of distinct powers of $1 + i$, and that the Gaussian integer z can be expressed as such a sum if and only if $i - z$ cannot.

12336. *Proposed by Szilárd András, Babeş-Bolyai University, Cluj-Napoca, Romania.* Let N be the center of the nine-point circle of triangle ABC, and let D, E, and F be the orthogonal projections of N onto the sides BC, CA, and AB, respectively. Prove that the Euler lines of triangles ABC, AEF, BFD, and CDE are concurrent. Prove also that the point of concurrency is equidistant from the circumcenters of AEF, BFD, and CDE.

12337. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For $k \in \{0, 1, 2\}$, let

$$S_k = \sum \frac{(-4)^n}{2n+1}\binom{2n}{n}^{-1},$$

where the sum is taken over all nonnegative integers n that are congruent to k modulo 3. Prove

(a) $S_0 = \dfrac{\ln\left(1+\sqrt{2}\right)}{3\sqrt{2}} + \dfrac{\pi}{6}$;

(b) $S_1 = \dfrac{\ln\left(1+\sqrt{2}\right)}{3\sqrt{2}} - \dfrac{\ln\left(2+\sqrt{3}\right)}{2\sqrt{3}} - \dfrac{\pi}{12}$; and

(c) $S_2 = \dfrac{\ln\left(1+\sqrt{2}\right)}{3\sqrt{2}} + \dfrac{\ln\left(2+\sqrt{3}\right)}{2\sqrt{3}} - \dfrac{\pi}{12}$.

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12338. *Proposed by István Mező, Nanjing, China.* Prove

$$\int_0^\infty \frac{\cos(x) - 1}{x\ (e^x - 1)}\,dx = \frac{1}{2}\ln\bigl(\pi \operatorname{csch}(\pi)\bigr).$$

12339. *Proposed by Cristian Chiser, Elena Cuza College, Craiova, Romania.* Let A and B be complex n-by-n matrices for which $A^2 + xB^2 = y\,(AB - BA)$, where x is a positive real number and y is a real number such that $(1/\pi)\cos^{-1}\bigl((y^2 - x)/(y^2 + x)\bigr)$ is irrational. Prove that $(AB - BA)^n$ is the zero matrix.

12340. *Proposed by Antonio Garcia, Strasbourg, France.* Let $g : [0, 1] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1 - x)^n} dx = Cg(1/2)$$

for some constant C (independent of g) and determine the value of C.

12341. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let $x_1, \dots, x_n$ be positive real numbers with $\sum_{i=1}^n x_i^2 \le n$, and let $S = \sum_{i=1}^n x_i$. Prove

$$\prod_{i=1}^n \left(1 + \frac{1}{x_i x_{i+1}}\right)^{x_i^2} \ge 2^{S^2/n},$$

where x_{n+1} is taken to be x_1.